PLANE GEOMETRY

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PLANE GEOMETRY

By

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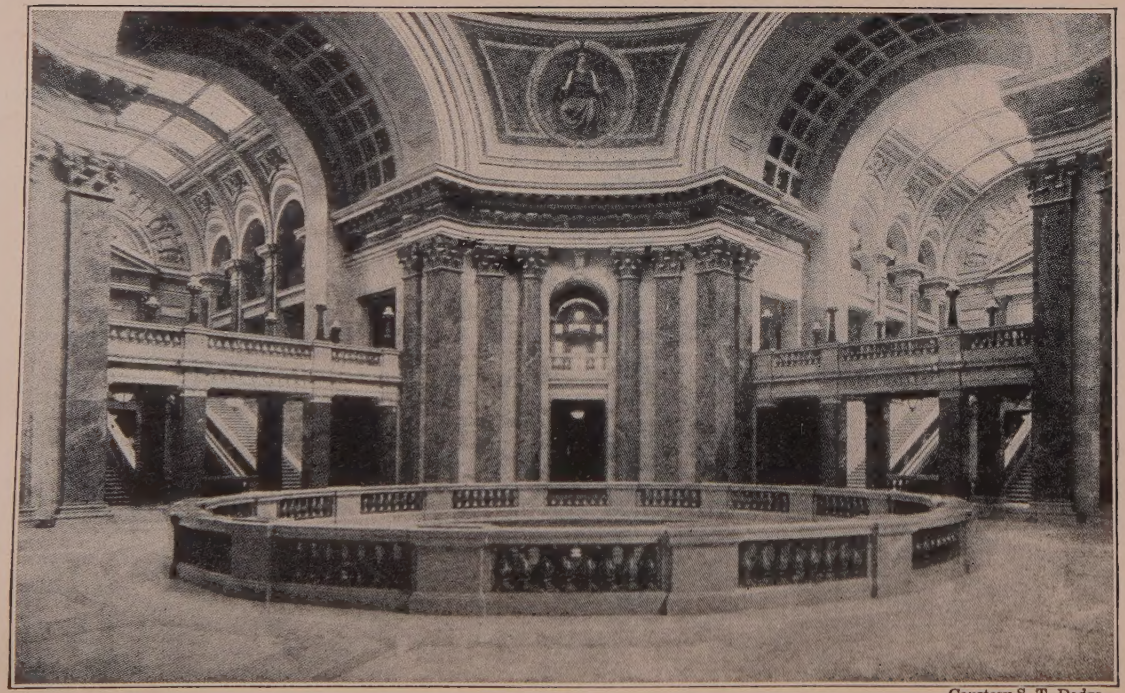
PLANE GEOMETRY



Courtesy S. T. Dodge

Looking into east and south stair halls from the gallery of the capitol at Madison, Wisconsin. The regularity of design in this building is an outstanding example of the use of geometry in architecture.

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THE PREFACE

The authors of this book are firmly convinced that those who study geometry should have a lasting impression of the beauty and usefulness of geometric forms and of the importance of geometric facts and should be able to do the easier parts of geometry for themselves.

Attention is called to the Introduction and to the previews given in connection with the various units and topics. These pages bring home to the student the widespread use of the forms he is to study.

Attention is also called to the following special features:

I. The analytical method of attack is employed in detail throughout.

There are, of course, many proofs that the student cannot reasonably be expected to think out for himself. In such cases the analysis gives him at the outset the gist of the argument, calls his attention to the method of proof employed, and gives him some idea of how the proof may have been originally thought out.

The student can and should make his own analyses for the easier work. To this end training is given by many detailed directions, as on pages 53–63, 275, and 278, and by sample analyses in the text.

II. The arrangement of the work is such as to throw emphasis on important theorems and methods.

Without emphasis neither effective analysis nor any lasting impression of the subject is possible. In this book, therefore, the division into units and topics is based on the big general ideas of geometry, and the arrangement within each division and subdivision is such as to call attention to fundamental theorems and methods.

The following are illustrations of the effect of proper emphasis:

a. The student will be able to make the analyses required in the later part of rectilinear figures because of the emphasis placed on fundamental facts and methods in the topic on congruent triangles in Unit II, Topic 1, and that on parallels and perpendiculars in Unit II, Topic 2.

b. In Units V and VIII the emphasis is such that the student gets clearly the big general ideas involved in the study of proportion and similarity. He learns in Unit V how to prove ratios and products equal. He gets in Unit VIII not only the tests for similar figures, but also their properties.

III. The traditional order is preserved except in Units V and VIII, as noted above. This change in order was made so that all properties of similar figures could be studied at the same time.

IV. The introduction of optional theorems and topics and the division of the exercises into C, B, and A lists make the text elastic enough to permit its use with classes of homogeneous grouping or for differentiated assignments within a group.

V. The division of the work into topics makes possible the effective use of the topic or unit method of presentation, if the teacher prefers this method. Unity is preserved in all units and topics.

VI. Summaries are given throughout the work where they will be most useful, and are followed by special review exercises. Survey tests are given for all work not marked optional. The survey tests and the C exercises after the summaries may be used for oral review. They have a distinct pedagogical value.

VII. The formal theory of limits is omitted.

The idea of a limit is presented informally in such exercises as Exercises 2 and 3 on page 78 and Exercise 23 on page 102. The treatment of the measurement of the circle will be found comprehensive and satisfying.

VIII. The various pictures included have in each case a direct bearing on the discussion in connection with which they occur.

IX. The section on geometric reasoning on pages 422–448 is unique. Much has been said of the value of the study of geometry in making students more careful reasoners and thinkers. This section calls attention to the fact that the thinking necessary to do good work in geometry is little different from other thinking and illustrates the fact by numerous examples from life.

The text is in accord with the recommendations of the Report of the National Committee on Mathematical Requirements and with the requirements of the state of New York and of the College Entrance Examination Board.

The first suggestion in recent geometry texts for the general use of an analysis or a plan of proof seems to have come from W. W. Hart of the University of Wisconsin. The authors wish to acknowledge their indebtedness to him in this particular.

Thanks are due many school friends for their criticisms and suggestions and to teachers in trial centers who have used the advanced pages in their classes.

M. S.
C. E. C.
C. M. A.

THE USE OF THE BOOK

To the teacher:

1. The analytical method should be used in all theorems and exercises and the student trained to make analyses.

2. A minimum course may be obtained by omitting all material marked "optional" and confining the exercises to the C lists and to exercises that immediately follow theorems and are simply called "exercises" in the text.

3. If enough easy exercises are not provided in this way, the teacher may choose from the B lists such exercises as will best meet the needs of her class at the time.

4. It is not necessary, or even desirable, that everything in the text be taken as the class proceeds.

5. The A exercises should be assigned with particular care. They are likely to be hard.

6. In assigning exercises from B and A lists care should be taken to see that theorems omitted are not involved.

7. It was not considered desirable to divide all sets of exercises into C, B, and A lists. If a topic is optional, there are no C exercises in connection with it.

8. The survey tests do not include optional work.

9. When preparing for the New York State Regents' examinations or for the examinations of the College Entrance Examination Board, work marked "optional" should be included.

10. Exercises in italics may be referred to in later work or are required in some state syllabi.

11. Material marked "optional" is optional with the teacher, not with the students.

PLANE GEOMETRY

INTRODUCTION: THE USES OF GEOMETRY

1. Geometry in life. Geometry, you may think as you open this book, is a subject entirely new to you. But is it? Before you decide, read these introductory paragraphs. You will see that you already know much about geometry, if not in name, at least in fact. The word *geometry* is derived from the two Greek words *ge*, "earth," and *metron*, "measure," which together mean "earth measurement." A mere definition, however, will give you little notion of the important part geometry plays in the world about you today, or of how vital it has been in the growth of civilization. If, as you study the subject, you will look about, you will be amazed at the variety of geometric forms you will discover. You will be amazed, too, at the numerous ways in which geometric principles are applied. You will find geometric designs used in your home and community, on the athletic field and the playground, in your subjects at school, in nature, art, architecture, and science, in manufacturing and industry, and in construction work. At intervals throughout this book illustrations will show how true this is.

2. In your home. Everywhere about you are examples of the figures that you are to study. Look through the line drawings, not the pictures, in this book and see whether you recognize any of them. Then think of your own home. Do you find any of these shapes there?

[1] *To the teacher:* This introductory material on the uses of geometry may or may not be studied first, according as you prefer. The ideas in this section may be drawn upon at points in the text where practical application can be made. The various sections may also be used for project work for individual students.

Look at the bricks in the wall or chimney, the window and door frames, and other things on the outside of the house. Inside the house look at the rugs or linoleum on the floors, the paper on the walls, the tiles in the bathroom, the coverings on the beds, the curtains at the windows, the draperies in the doorways, the materials of which clothing is made, and little things that people use, such as compacts, purses, and cigarette lighters.

Ex. 1. Bring into class a list of at least five things you find in your home that illustrate a geometry figure in shape or design. Get as many different geometric shapes as possible. Be definite. Tell where each pattern is found and if possible the name of the geometric figure that is illustrated.

Ex. 2. Bring into class five or more pictures each showing a different thing from the home that illustrates a geometric figure. Your pictures should illustrate at least four different geometric shapes. You will find these largely in advertisements, such as those in newspapers or magazines, in mail order catalogs or catalogs of firms making or selling linoleum, leather goods, grilles, wall paper, and the like. Cut your pictures out neatly and paste them on sheets of notebook paper. You may bring a sample of cloth instead of one picture if you like. On each picture indicate clearly the figure that you think is geometric and give its name. If you do not know the names of the various shapes, ask your teacher.

3. In your community. Look about you in your home town or city and see whether you can discover any evidences of the uses of geometry. Notice the flower beds in the parks or in private gardens, the layout of sidewalks and of street-car tracks, the shapes of the concrete blocks forming the sidewalks, and elevated train structures.

Ex. 1. Bring into class a list of at least three such uses of geometry that you find in your community. In each case name the geometric form and if possible the purposes it serves.

Courtesy Chamber of Commerce, Salt Lake City

FIG. 1. **Salt Lake City from the air**

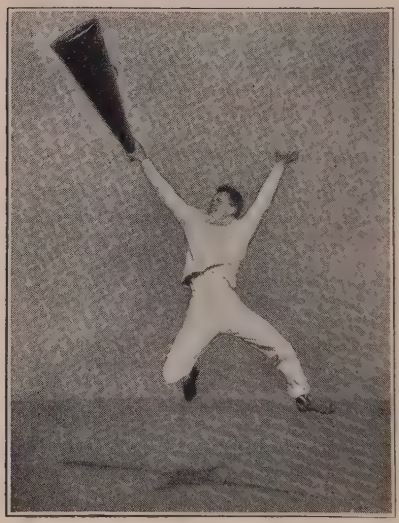
Fig. 1 shows an airplane view of Salt Lake City. Notice the manner in which the streets are laid out.

Ex. 2. Do you know of any other city or town that is laid out in this way? Do the streets ever bear any particular relation to the points of the compass? Is there any advantage in laying out streets in this way? Do you know any town or city where the streets are not laid out in this way? Are there any particular disadvantages in this case?

Ex. 3. Bring into class pictures of the uses of geometry in the community. Paste them on sheets of notebook paper and name the geometric figures illustrated.

4. In relation to play. Study the various games and forms of play that you know for evidences of geometric forms. This study should include the marks the younger children make on the sidewalks in their games and any

markings on their toys. Notice also the layout of the grandstand in the football field, the football field itself, the baseball diamond and tennis court; chess, checker, backgammon, and parchesi boards; pieces of apparatus in public playgrounds and school gymnasiums and the marks on the gymnasium floor.



Acme Photo

FIG. 2. **Is there any geometry here?** The megaphone is modeled on one of the basic figures of solid geometry—the cone.

Exercise. Write up the results of your study for your notebook, putting in at least three pictures to illustrate what you have to say.

5. In geography. The meridians and parallels of latitude are geometric lines used for ease in determining the positions of places on the earth. If you glance at the outline maps of two states, such as Virginia and Iowa, you will notice that the counties in one are very different in shape from those in the other. When the United States wished to lay out the public lands west of the Alleghenies for sale, Congress passed a law that they should be laid out in squares a mile on each side with lines running east and west and north and south. This could not be done exactly. Can you think of any reason why? How the surveyor general carried out this law you will learn if you ever study surveying. In general, county and township lines follow the lines of

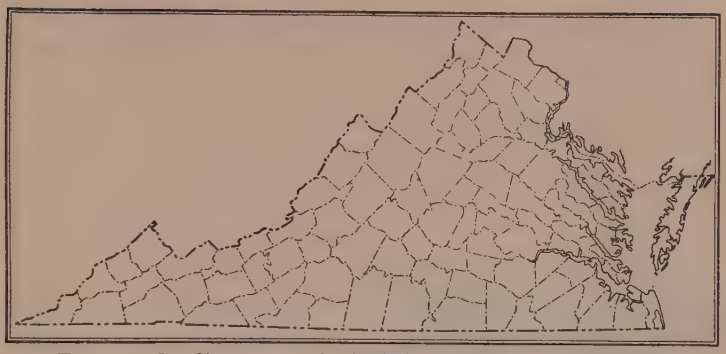


FIG. 3. **Outline map of Virginia showing county lines**

the public surveys. The outline county map of Iowa is typical of this arrangement. Many of the roads in western states are laid out along county and township lines. This is not true in the East. Most maps are drawn to scale and illustrate the use of ratio and proportion. The ratio of lines is studied in geometry.

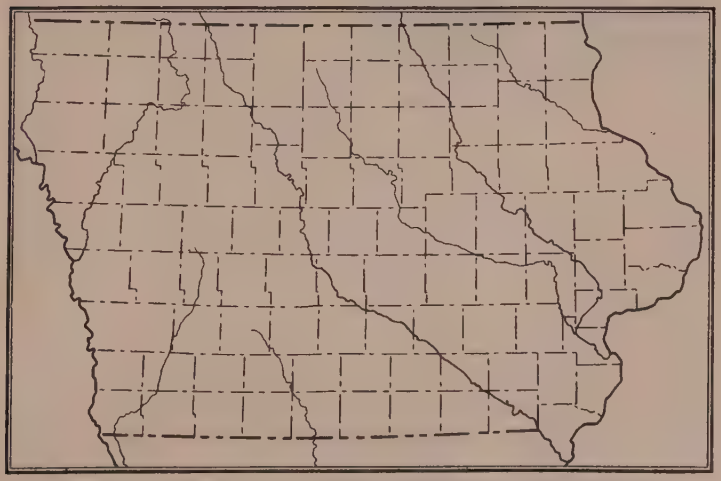


FIG. 4. **Outline map of Iowa showing county lines**

Ex. 1. Find in some book on surveying a diagram showing how townships are laid out and how they are divided into sections a mile on each side.

Ex. 2. Can you find any illustrations of geometric figures in land forms, mountains, lakes, or streams?

6. In nature. There are many illustrations of geometric forms to be found in nature. The trunks of trees are circular, but the cross sections of the stems of some other plants are triangles or squares. Petals are arranged at equal intervals around flowers.

Courtesy United States Weather Bureau

FIG. 5. **Hailstones**

Ex. 1. Cut an apple across the core at right angles to the stem. Make a drawing of the section of the core and name the geometric figure which it resembles.

The comb of bees (see Fig. 534, page 371) and the cells in the wasp's nest are six-sided. The most common form of the starfish is a regular five-side (Fig. 536, page 373). Mineral crystals furnish a striking illustration of geometric forms. Snowflakes are found in numberless beautiful patterns (Fig. 162, page 114). The cross sections of hailstones are interesting.

Ex. 2. Find in the encyclopedia or some book on crystals, the shapes of the faces of various crystals. Make drawings of some of these forms and as far as possible name them. Put these on sheets of notebook paper.

Ex. 3. Bring into class two other pictures of geometric forms in nature and paste them on sheets of notebook paper.

7. In art. Geometric forms are very common in certain kinds of industrial ornament such as grilles, registers, tiles, inlaid linoleum, and parquet floors.

Tiles are in standard sizes all of which are obtained from the six-inch square.

Ex. 1. What is an encaustic tile? Must the design necessarily be geometric?

Inlaid linoleum is made of pieces of different colors glued to a canvas backing and welded into one whole.

Ex. 2. What is the difference between the patterns on inlaid linoleum and those on ordinary oilcloth? Why?

Ex. 3. See whether you can learn how parquet floors are laid. Why are the designs geometric?

Fig. 147, page 105, shows an Indian lattice pattern based on a network of equilateral triangles. Mohammedan art is almost entirely geometric, because of certain religious restrictions. Patterns like Fig. 147 were used for wall decorations where they were executed in plaster or on tiles or for window lattices instead of glass. Occasionally one sees such patterns on modern buildings in this country. Geometric designs on tiles and mosaic floors date back to the old Romans.

The designs of primitive peoples are often geometric (see Fig. 241, page 160).

Ex. 4. Find illustrations in this book of the uses of geometric designs in ornament. Bring into class pictures of such designs not in this book. Paste them on sheets of notebook paper and mark clearly the shapes illustrated.

8. In architecture. The first arches used were made by the Romans and were semicircular (see Fig. 268, page 190). Later, arches of various shapes made of parts of two or more circles were used in European cathedrals and Mohammedan mosques and other buildings. The

arches in the best modern buildings are curves that you will study in higher mathematics. The cross sections of moldings are often curves made of parts of two or more circles (see Fig. 304, page 218), while the designs of many church windows are based on triangles and circles. Turn to pages 214, 215, and 219 and see whether you can find any such designs. Gratings and grilles are often of geometric design (see pages 240 and 272). The trusses that support the roofs of buildings are based on geometric figures. A large percentage of them are triangular (Fig. 95, page 68; page 69; Ex. 10, page 77; Fig. 256, page 175). Decorated rafters often seen in churches and assembly halls are geometric in shape and are decorated with geometric openings (Figs. 296 and 298, page 214).

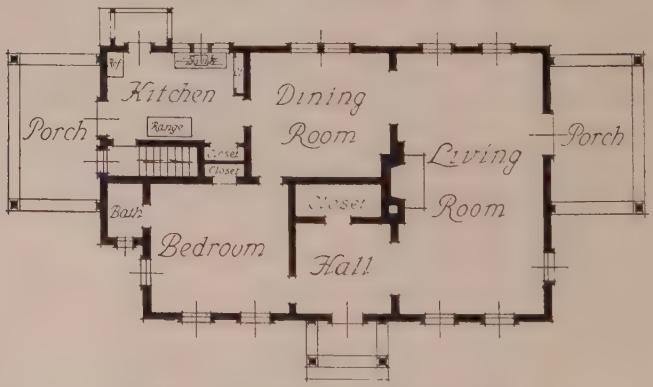
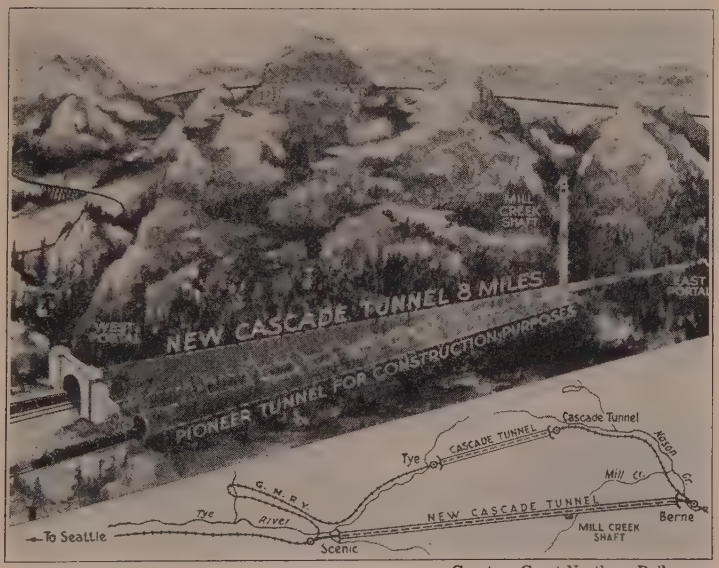


FIG. 6. **An architect's floor plan**

Floor plans such as one sees in blueprints and architects' drawings are drawn to scale by ratio.

Exercise. Bring into class five pictures or diagrams that you have found illustrating the use of geometry in architecture. You will find commercial catalogs useful. Paste your pictures neatly on sheets of notebook paper and indicate in each case the geometric shape used.

9. In engineering. In building a railroad through mountainous country, it is necessary to make use of horizontal and vertical curves, to get around and over mountains. These curves must be laid out by geometric methods (Fig. 7). Sometimes it is necessary to go through a mountain by a tunnel which also requires very careful and accurate geometric planning.



Courtesy Great Northern Railway

FIG. 7. **The new Cascade Tunnel and the old line abandoned January 12, 1921.** The engineering problems involved in the construction of the new tunnel through the Stevens Pass in the Cascade Mountains were extremely difficult ones.

The main tunnel as completed is 7.79 miles long; it is absolutely straight from end to end, and is on a grade of 1.56 descending from east to west. The calculations were made and the instrument work was performed with remarkable accuracy by the engineers. When the openings from the east and west ends met, 3,000 feet underground and 4 miles from the west portal, they were only 7 inches apart, with a difference in elevation of only 9 inches, and a difference in calculated and actual distance of only 12 inches!

Courtesy J. W. Clark

FIG. 8. **A village near the Gizeh Pyramids, not far from Cairo, Egypt.** The triangular outline of the faces of the pyramids in the distance is easily discernible. If you were near these pyramids, you would find that each has a square base and four triangular faces which once met at an apex.

J. A. Glenn, Albany, New York

FIG. 9. **The Gorge of the Niagara River and the Whirlpool Rapids.** Observe the perpendicular and parallel lines with the triangular bracing faintly shown in the bridge.

In construction work, from the Pyramids of Egypt to the Empire State Building in New York, from the Roman aqueducts to the Roosevelt Dam or the long spans of modern bridges, geometric principles must be followed.

10. Miscellaneous. Geometric forms and principles are used in many places besides those mentioned above.

Cut-glass designs are largely geometric.

The farmer plants his corn or his orchard and many other crops in parallel rows to make cultivation of the soil and harvesting easier.

Ex. 1. Are all crops planted in geometric patterns? Why?

Ex. 2. What geometric figure can you find in connection with garden hose? machine nuts?

Ex. 3. What kind of line does the cutting tool on a lathe make?

Pulleys and belts seen in machine shops illustrate certain facts that you will study when you get to circles.

Courtesy California Promotion Committee, San Francisco

FIG. 10. **A citrus fruit orchard in California**

Many facts that you will study in solid geometry are of fundamental importance in astronomy and navigation.

11. Flatland. Most of the objects to which we have called your attention in the preceding pages occupy space; they have length, breadth, and thickness. Forms that have three dimensions are studied in solid geometry. Consider some of them for a moment.

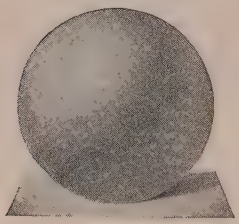


FIG. 11

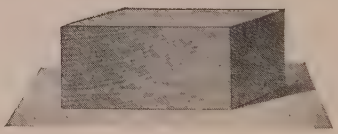


FIG. 12

FIG. 13

FIG. 14

A rectangular block has faces called surfaces, edges called lines, and corners called points. The cylinder has no corners. The sphere has neither edges nor corners. The edges of a cone and cylinder are not like those of the rectangular block.

The surface of each of the four solids is different from that of each of the others. We can make drawings on

each of these surfaces. The rail of a spiral staircase is a line drawn on a cylinder. The figures drawn on a sphere are studied in solid geometry. In this book you are to study some figures drawn on a flat surface like the faces of the rectangular block. Only two dimensions are necessary. Thus you will see why the subject you are to study is called *plane* geometry. You might, in other words, say that you will work and dwell in flatland.

12. Origin of geometry. Geometry is very old. It was used by the ancient Egyptians about 3000 B.C., in building the pyramids and in surveying farm lands in the Nile Valley. As they did not prove the rules that they used, they made many mistakes. Later the Greeks as early as 600 B.C., or before, began to prove the facts studied in geometry. They have really given us the geometry that we study today.

Exercise. Look up, in some history of mathematics, stories concerning Thales, Plato, Pythagoras, Euclid, and Archimedes and their geometry work.

The men mentioned in the exercise above were Greeks who loved geometry.

13. A look ahead. You have some idea from the work above of the shapes and figures with which you are to work. You are to find the relationships between these figures and the special properties that make them useful and interesting. In studying geometry it is necessary to be able to read your text carefully and to think clearly. If you do this, geometry is fun; more fun by a long way than any newspaper puzzle. But it takes time and thought to get the best there is out of your work.

14. Abbreviations and Symbols Used

Fig. . . . . figure
Ex. . . . . exercise
Exs. . . . . exercises
Def. . . . . definition
Ax. . . . . axiom
Axs. . . . . axioms
Th. . . . . theorem
Ths. . . . . theorems
Cor. . . . . corollary
alt. int. . . alternate interior
comp. . . . complementary
const. . . . construction
sup. . . . . supplementary
adj. . . . . adjacent
§ . . . . . section
$\therefore$ . . . . . therefore
° . . . . . degree
′ . . . . . minute
″ . . . . . second
cm. . . . . centimeter
mm. . . . . millimeter
$=$ . . . . . equal to, is equal to
$\cong$ . . . . . congruent to, is congruent to
$\sim$ . . . . . similar to, is similar to
$>$ . . . . . greater than, is greater than
$<$ . . . . . less than, is less than
$\angle$, $\measuredangle$ . . . . angle, angles
rt. $\angle$ . . . . right angle
rt. $\measuredangle$ . . . . right angles
$\perp$ . . . . . perpendicular, perpendicular to, is perpendicular to
$\perp s$ . . . . . perpendiculars
$\parallel$ . . . . . parallel, parallel to, is parallel to
$\triangle$, $\triangle s$ . . triangle, triangles
rt. $\triangle$ . . . right triangle
rect. $\square$. rectangle
rects. $\square s$. rectangles
sq. . . . . square
sqs. . . . . squares
$\square$, $\square s$. parallelogram, parallelograms
trap. . . . trapezoid
traps. . . . trapezoids
$\odot$, $\odot s$ . . . circle, circles
$\overset{\frown}{AB}$ . . . arc AB
$\overline{AB}$ . . . . . chord AB
$\overline{AB}^2$ . . . the square of the segment AB
A . . . . . area
per. . . . . perimeter
Post. . . . postulate
Posts. . . postulates

UNIT I

GENERAL DEFINITIONS AND PRINCIPLES

Points and Straight Lines

15. Starting with what you already know. You may be familiar in a general way with some of the material treated in this unit. To be able to do more complicated work later, it is essential that you understand simple fundamental facts and principles. They are therefore given at the beginning of your work. Notice the methods given for naming lines, points, and angles. They are probably new to you.

16. A point is represented by a dot made with a sharp-pointed pencil. It is read by a capital letter placed near the dot (see point A, Fig. 15).

17. A straight line is represented by a mark made by a sharp-pointed pencil and a ruler. It may be read by a small letter placed near the mark (see line n, Fig. 15).

Fig. 15

18. A straight line resembles a tightly stretched string, such as a thread by which a weight is suspended. The word *line* comes from the Latin word *linea*, meaning "a linen thread." The word *straight* comes from the Anglo-Saxon verb meaning "to stretch."

19. If you put two points on a piece of paper and lay your ruler on them, you will see that you can draw a straight line through these two points. But you can draw only one straight line through them. If, therefore, you know that a straight line passes through two given points, you may read the line by these two points, as line AB (Fig. 16).

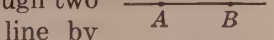


Fig. 16

It must be remembered that a line extends indefinitely in each direction. It has no ends. The points A and B are anywhere on the line. If three points are known to lie on a line, the line may be read by any two of these points.

C Exercises†

1. In Fig. 17 read the following lines by two capital letters: line k, line l, line n, line m.

2. In Fig. 17 read the following lines by one small letter: line AC, line AE, line EC, line BC, line DB, line DC.

3. At what points do lines k and n cross, or intersect? lines l and n? EC and BF? k and AF (Fig. 17)?

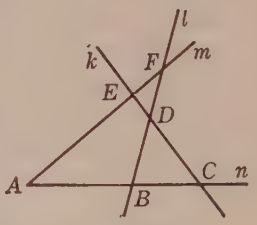


FIG. 17

Draw figures to illustrate questions 4–6. Letter your figures and read the lines by the most convenient method.

4. How many straight lines can be drawn through (*a*) one given point? (*b*) any two given points? (*c*) any three given points? (*d*) any four given points?

5. How many points are there in a straight line?

6. What is the greatest number of points in which two straight lines can intersect? in which three lines can intersect?

B Exercises†

1. Draw three different straight lines with (*a*) no possible intersections; (*b*) only one possible intersection; (*c*) only two possible intersections; (*d*) three intersections.

2. Can you draw four different straight lines so that there will be (*a*) no possible intersections? (*b*) only one possible intersection? (*c*) only two possible intersections? (*d*) only three possible intersections? (*e*) only four possible intersections? Can you get more than four possible intersections?

† *To the teacher:* Exercises not marked and C Exercises should be required of the entire class. A and B Exercises are optional. A Exercises are for honor or superior students only.

20. Rays. A part of a straight line that starts at a given point and extends indefinitely in one direction only is called a **ray.** The point from which the ray starts is called the **origin** of the ray (see ray a, origin O, Fig. 18).

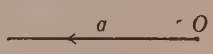


FIG. 18

Exercise. Find illustrations of rays in Fig. 17. What is the origin of each? Turn to figures elsewhere in the book and find illustrations of rays.

21. Segments. A part of a straight line between two given points on the line is called a **straight-line segment.** Hereafter the single word **segment** will be used when a straight-line segment is meant. In Fig. 17 segment BC is part of line n.

Exercise. Find twelve segments in Fig. 17 and name the line of which each is a part.

22. Notice that a line has no ends, a ray has one end, and a segment has two ends.

23. A **segment may be transferred** from one line to another by the use of the compasses, the dividers, or a strip of paper. If the compasses or the dividers are used, open them to the segment required by laying them on the given segment, as segment AB, line n (Fig. 19). Without changing the adjustment, place one leg on point C and mark off segment AB on line l with the other leg.

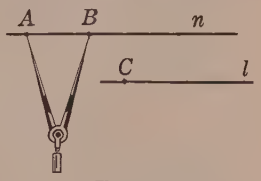


FIG. 19

24. Two **segments** may be **added** by placing them end to end on the same line. Thus in Fig. 17 segment AC is the sum of segments AB and BC. In other words, $AB+BC=AC$.

Exercise. In Fig. 17 find three other illustrations of a segment that is the sum of two segments. Write the statement expressing the relation as given above for segment AC.

25. A smaller segment may be **subtracted** from a larger segment. Place the smaller segment on top of the larger, with an end point of one on an end point of the other; thus in Fig. 20 the segment BC is the difference between segments AC and AB. In other words, $AC - AB = BC$.

FIG. 20

Ex. 1. In Fig. 17 find seven other illustrations of a segment that is the difference between two segments. Write the statement expressing the relation as given above for segment BC.

Ex. 2. Draw a figure of your own made up of five or six lines crossing in different ways. In your figure find at least five illustrations of a segment that is the sum of two or more segments and at least five illustrations of a segment that is the difference between two segments.

Ex. 3. Find elsewhere in the book five figures that contain at least one illustration of a segment that is the sum of two segments and of a segment that is the difference between two segments. In each case write the statement expressing the relation as explained in §24 and §25.

26. A segment has a definite **length** in some standard unit such as centimeters, feet, or inches. Two segments are said to be **equal** if they have the same length when measured in the same unit.

27. The length of a segment joining two given points is often called the **distance** between the two given points.

28. Two segments are said to be **congruent** if they can be placed upon each other so that the end points of one will fall upon the end points of the other. It is evident that congruent segments are equal and that equal segments are congruent.

Exercise. Draw any segment on a piece of paper. Show by folding the paper how to get a segment that is half as long as the given segment; that is, bisect the given segment.

Fundamental Geometric Assumptions

29. Fundamental principles that are accepted as true without proof are called, in general, assumptions. In this book they are called **postulates** when they refer to geometric facts. The truth of the following statements should be evident from the discussion on the preceding pages. Draw a figure to illustrate each of the following:

Post. 1. An indefinite number of straight lines can be drawn through one point.

Post. 2. Only one straight line can be drawn through two given points; that is, two given points locate a straight line definitely or determine a straight line.

Post. 3. Two different straight lines can intersect in only one point; that is, two intersecting straight lines locate or determine a point.

Post. 4. Only one segment can be drawn between two given points; that is, a segment is determined in position and in length if its end points are known.

Post. 5. A segment can have one and only one mid-point.

Post. 6. Only one ray can be drawn from a given origin and pass through a second given point.

Angles

30. A figure formed by two rays which have the same origin is called an **angle**. The rays are the **sides** or **arms** of the angle, and the origin of the rays is its **vertex**.

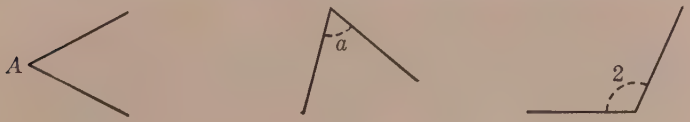


Fig. 21

31. An angle may be read in various ways. In Fig. 21, p. 19, the angles are, from left to right: $\angle A$, $\angle a$, $\angle 2$.

An angle may be **read by three letters.** Notice that in this case the letter at the vertex of the angle is read between the other two letters; as in Fig. 22 the angle is $\angle CAB$ or $\angle BAC$.

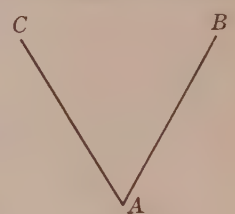


FIG. 22

Notice in Fig. 23 that $\angle 1$ may be read by three letters in various ways: $\angle 1$ is $\angle BAF$, or $\angle EAB$, or $\angle CAE$, or $\angle BAE$. All that is necessary is that the vertex letter be read between letters that are somewhere on the sides of the angle. Can $\angle 1$ be read in any other way?

Ex. 1. Read the vertex and the sides of each of the following angles: $\angle 5$, $\angle r$, $\angle w$, $\angle y$, $\angle FBA$, $\angle 6$, $\angle 9$, $\angle ODB$ (Fig. 23).

Ex. 2. In Fig. 23 read by three letters each of the angles marked by a number or by a small letter.

Ex. 3. Read by a number or small letter each of the following: $\angle DBO$, $\angle ODE$, $\angle BFO$, $\angle BAF$, $\angle DEB$, $\angle BDC$, $\angle FBO$ (Fig. 23).

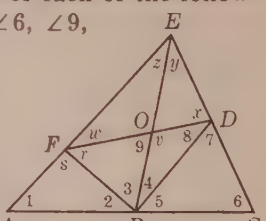


FIG. 23

Ex. 4. Find elsewhere in the book figures that show illustrations of angles that can be read in more than one way. In each case give the vertex and the sides of the angle.

32. Congruent angles. Exercise. Draw an angle on a piece of paper with a soft pencil. Put it face down on another piece of paper and rub with the flat part of your thumb nail. You have traced the angle. The original angle and its trace are congruent.

Two angles are said to be **congruent** if they can be so placed that their sides will fall along each other.

NOTE. The word *congruent* is applied to any two figures that can be made to coincide.

33. Adjacent angles. Angles that have a common vertex and a common side which separates the angles are called **adjacent angles.**

Ex. 1. In Fig. 24 are $\measuredangle 1$ and 2 adjacent? Why? $\measuredangle 2$ and 3? Why? $\measuredangle 3$ and 4? Why? $\measuredangle 2$ and 4? Why?

Ex. 2. In Fig. 23 point out seven pairs of adjacent angles. In each case name the common vertex and common side.

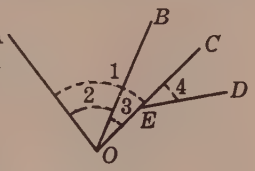


FIG. 24

34. Ex. 1. Draw on different pieces of paper two angles that are not congruent. Mark them $\angle 1$ and $\angle 2$. Hold them to the light and arrange them so that they are adjacent. Trace one onto the other. Mark the sum of the two angles, $\angle s$.

If two angles are adjacent, an angle is formed which is the **sum of the two adjacent angles**; thus, in Fig. 23, $\angle ABO$ is the sum of $\angle 2$ and $\angle 3$. Therefore, **to add two angles geometrically, place them adjacent.**

Ex. 2. In Fig. 23 find six illustrations of an angle that is the sum of two or more angles. Make a definite statement in each case, thus: $\angle ABO = \angle 2 + \angle 3$.

35. Ex. 1. Draw on different pieces of paper two angles that are not congruent. Mark them $\angle 1$ and $\angle 2$. Hold them to the light and arrange them so as to show the difference when the smaller angle is subtracted from the larger one. Mark the difference between the two angles $\angle d$.

A smaller angle may be **subtracted** from a larger. Place the smaller angle inside the larger so that they have a common vertex and a common side. An angle is formed which is the **difference** between the two angles. In Fig. 23, $\angle 8$ is the difference between $\angle ODC$ and $\angle 7$.

Ex. 2. In Fig. 23 find six illustrations of an angle that is the difference between two angles. Make a definite statement in each case, thus: $\angle ODC - \angle 7 = \angle 8$.

36. Right angles and perpendiculars. Exercise. Draw a line DA. Take any point in DA and mark it B. Fold the paper so that the ray BA will fall on the ray BD. Crease the paper sharply. Why are the two angles formed congruent and adjacent? Why is the crease perpendicular to DA?

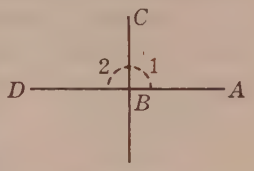


FIG. 25

When one straight line passes through a point in another and forms two congruent adjacent angles, the angles are called **right angles** and the lines are said to be **perpendicular**. In Fig. 25, $\angle 1$ and $\angle 2$ are right angles. Why? CB is said to be perpendicular to DA. Why?

37. The length of the perpendicular from a point to a line is called the **distance from the point to the line.**

38. An **angle** may be considered as **formed by the rotation of a ray about its origin.** The size of the angle depends upon the amount of rotation. In Fig. 26 which is the larger angle? Why?

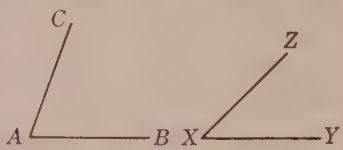


FIG. 26

39. In Fig. 25 suppose the ray BA to rotate about point B as origin. One complete rotation would bring it back to position BA. A right angle is obtained by one-quarter of a complete rotation.

40. In Fig. 27 the ray OA may start from the position OA and rotate until it extends in a direction directly opposite to its original position. It has made one-half of a complete rotation. An angle made by one-half of a complete rotation is called a **straight angle** (Fig. 27, $\angle AOA_5$).

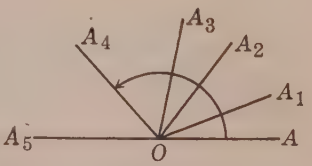


FIG. 27

41. An angle less than a right angle is called an **acute angle** (Fig. 27, $\angle AOA_1$).

42. An angle greater than a right angle and less than a straight angle is called an **obtuse angle** (Fig. 27, $\angle AOA_4$).

43. An angle greater than a straight angle, but less than one complete rotation, an angle made by one complete rotation, and angles made by more than one complete rotation are used in higher mathematics. They have very little place in your work.

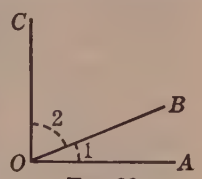


FIG. 28

44. Two angles are called **complementary** if their sum is a right angle. In this case each angle is called the **complement** of the other. Find in Fig. 28 two complementary angles. What angle is the complement of $\angle 1$? of $\angle 2$? How do you obtain the complement of an angle?

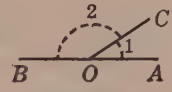


FIG. 29

45. Two angles are called **supplementary** if their sum is two right angles or a straight angle. In this case each angle is called the **supplement** of the other. Find in Fig. 29 two supplementary angles. What angle is the supplement of $\angle 1$? of $\angle 2$? How do you obtain the supplement of an angle?

46. Two angles are called **vertical** or **opposite** if the sides of one are prolongations of the sides of the other. If two lines intersect, two pairs of vertical angles are formed. Find in Fig. 30 two pairs of vertical angles.

FIG. 30

Exercise. Find elsewhere in the book figures containing supplementary angles, complementary angles, and vertical angles.

47. To measure an angle, you must find how many times it contains some other angle considered as a unit.

Sometimes one measures angles by comparing them with a right angle. Thus an angle may be $\frac{1}{2}$ of a right angle; $\frac{2}{3}$ of a right angle; $\frac{4}{3}$ of a right angle.

48. A good practical unit for measuring angles is $\frac{1}{90}$ of a right angle or $\frac{1}{360}$ of a complete rotation and is called a **degree** (°). Each degree may be divided into 60 congruent parts called **minutes** (′). Each minute may be divided into 60 congruent parts called **seconds** (″).

49. The number that tells how many times the unit angle is contained in a given angle is the **measure** or the **measure number** of the given angle. Angles that have the same measure number are said to be **equal.** Since congruent angles can be so placed that their sides take the same direction, congruent angles represent the same amount of rotation and have the same measure number. Also angles that have the same measure number represent the same amount of rotation and can be made to coincide. In other words, **equal angles are congruent and congruent angles are equal.** This should be remembered whenever equal angles are mentioned.

50. A **protractor** is an instrument for measuring angles. It is used like any other scale for measuring. The form of protractor shown in Fig. 31 is marked from 0° to 180° in both directions. It is also marked to show the supplements of angles.

Ex. 1. To measure an angle with a protractor (Fig. 31):

Place the protractor with the center O on the vertex of the angle and the line of zeros AO along one side of the angle. Read off the number of degrees on the scale as indicated by the other side of the angle. Extend the sides of the angle if necessary.

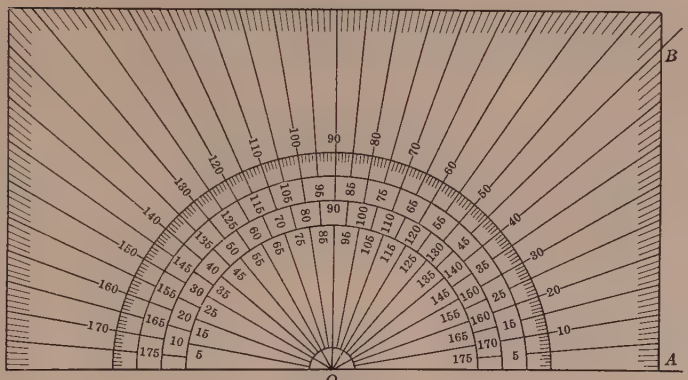


FIG. 31

Ex. 2. Show how to construct with the protractor an angle that shall have a given number of degrees.

C Exercises

1. How many degrees in a straight angle? in $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{1}{8}$ of a straight angle?

2. How many degrees in $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{4}{3}$, $\frac{4}{5}$ of a right angle?

3. How many degrees in the complement of:

a. 67°	*c*. 41° 10′	*e*. 31° 29′	*g*. 72° 21′
b. 49°	*d*. 53° 42′	*f*. 25° 32′	*h*. 18° 50′

4. How many degrees in the supplement of:

a. 64°	*c*. 14° 35′	*e*. 27° 65′	*g*. 113° 43′
b. 73°	*d*. 137° 50′	*f*. 142° 22′	*h*. 98° 2′

5. Define straight, acute, and obtuse angle. Make a drawing to illustrate each. Letter each drawing so that your angle may be read (*a*) by three letters, (*b*) by a number, or (*c*) by a single letter.

6. Define adjacent angles, supplementary angles, complementary angles, and vertical angles. Make a drawing to illustrate each and mark as in Ex. 5.

7. Draw an acute angle. Letter its vertex O and its sides OA and OB. Read it by three letters. Mark it $\angle 1$. Draw its supplement. Mark the supplement $\angle s$. Fold the paper so as to show the complement of $\angle 1$. Mark the complement $\angle c$.

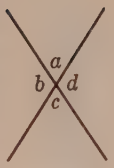


FIG. 32

8. In Fig. 32 show that $\angle b$ and $\angle d$ are supplements of the same angle. Show that $\angle a$ and $\angle c$ are supplements of the same angle.

9. Find in Fig. 33 as many illustrations as possible of the kinds of angles mentioned in Exs. 5 and 6, p. 25. The lines that are heavy at the intersections are perpendicular to each other.

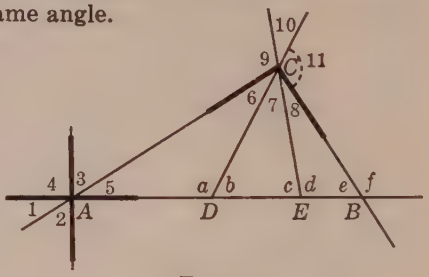


FIG. 33

10. Find the number of degrees in the angles named below (Fig. 33):

a. $\angle 1$ and $\angle 3$, if $\angle 5 = 36°$. *b*. $\angle f$, if $\angle e = 40°$.

c. $\angle 8$ and $\angle 11$, if $\angle 6 = 28°$ and $\angle 7 = 34°$.

11. Find in Fig. 33 a case of angles that are supplements of the same angle; complements of equal angles.

12. Find the number of degrees in the angles named below (Fig. 33):

a. $\angle 2$, $\angle 3$, and $\angle 5$, if $\angle 1 = 26°$.

b. $\angle 7$, $\angle 9$, and $\angle 11$, if $\angle 6 = 25°$ and $\angle 8 = 20°$.

c. $\angle 6$, $\angle 11$, $\angle 10$, and $\angle 9$, if $\angle 7 = 40°$ and $\angle 8 = 28°$.

13. Draw a line AB. Take O any point in the line. Show how to fold the paper so that the crease will be perpendicular to AB and will pass through point O.

14. Answer question 13 but take point O outside AB.

15. Draw an $\angle AOB$. Show how to fold the paper so that the crease will divide the angle into two equal parts, that is, so that the crease will bisect the angle.

16. Show how to answer questions 13 and 14 by using a rectangular card or a draftsman's triangle.

Exercises B†

NOTE. In the construction exercises that follow, perpendiculars may be drawn by paper folding or may be copied by use of a rectangular card or a draftsman's triangle. An angle may be made equal to a given angle by tracing (see §32) or by use of an angle-carrier. An angle-carrier is made by fastening together two pieces of wood or cardboard or two celluloid rulers (Fig. 34). On the blackboard the ray that bisects an angle will have to be drawn by the eye. Constructions with ruler and compasses are given later.

FIG. 34

Draw two unequal segments. Mark the larger one a and the smaller one b. Construct the following and mark all results clearly:

1. $a+b$. **2.** $a-b$. **3.** One-half the sum of a and b.

4. One-half the difference of a and b.

5. The sum of one-half of a and all of b.

6. One half of $2a+b$.

Draw two unequal angles. Make the smaller angle acute and mark it, $\angle y$. Mark the larger angle, $\angle x$. Construct the following (Exs. 7–12) and mark all results clearly:

7. $\angle x+\angle y$. **8.** $\angle x-\angle y$.

9. One-half the difference of $\angle x$ and $\angle y$.

10. One-half the supplement of $\angle x$.

11. One-half the complement of $\angle y$.

12. One-half the supplement of $\angle y$.

13. What kind of angle is the complement of an acute angle?

14. What kind of angle is the supplement of an obtuse angle?

15. The sum of three angles is 360°. The second is three times the first and the third is four times the first. Find the number of degrees in each.

16. If the complement of an angle is $\frac{1}{4}$ of the angle, find the number of degrees in the angle and in its complement.

† *To the teacher:* Any, all, or none of these exercises may be used, as you desire.

17. An angle is $\frac{2}{3}$ of its supplement. Find the number of degrees in the angle and in its supplement.

Find the number of degrees in an angle if:

18. The angle is 40° 15′ larger than its supplement.

19. The complement is 25° smaller than the angle.

20. Angle ABC is 40°. Sides AB and BC are each two inches long. If the sides were extended so that they measured four inches, how large would angle ABC be then?

21. How many degrees in the supplement of $\angle x$ if its complement is 23°?

22. How many degrees in the complement of $\angle x$ if its supplement is 133°?

A Exercises

By the methods indicated in B Exercises, page 27, construct the following with $\angle x$ greater than $\angle y$:

1. The supplement of one-half of $\angle x$.

2. The complement of one-half of $\angle y$.

3. Construct an angle of 45°; of $22\frac{1}{2}$°.

4. Show that the supplement of $\angle y$ may have two different positions with respect to $\angle y$.

5. What kind of angles are formed by the two supplements obtained in answer to question 4?

6. Show that the complement of $\angle y$ may have two different positions with respect to $\angle y$.

7. Draw two complementary adjacent angles and the bisector of each. How many degrees in the angle made by the bisectors? Why?

8. Answer question 7 for two supplementary adjacent angles.

9. Find on separate pieces of paper the supplement and the complement of $\angle y$. By tracing subtract the complement from the supplement. Explain your result. Can you work this exercise by algebra?

10. If a rafter of a roof of a barn starts up from the loft floor at an angle of 30°, what angle does the rafter make with the side of the house? If the wall were carried a little higher for a signboard, what angle would the signboard make with the roof?

General Assumptions

51. Certain laws that have been assumed and used in algebra are also assumed and used in geometry. In this book they are called **axioms.**

Ax. 1. If equals are added to equals, the results are equal.

In algebra:

If $x+y=5$
and $x-y=2$
then $2x=7$

In geometry:

If $\angle x = \angle y$ (Fig. 35)
and $\angle z = \angle w$
$\angle x + \angle z = \angle y + \angle w$
or $\angle ABC = \angle DEF$

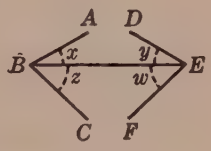


FIG. 35

Ax. 2. If equals are subtracted from equals, the results are equal.

In algebra:

If $x+y=5$
and $x-y=2$
then $2y=3$

In geometry:

If $AB=XZ$ (Fig. 36)
and $AC=YZ$
then $AB-AC=XZ-YZ$
or $BC=XY$

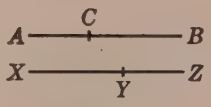


FIG. 36

Ax. 3. Doubles of equals are equal, or, more generally, if equals are multiplied by equals, the results are equal.

If $AC = XZ$ (Fig. 37)
and $2AC = AB$
and $2XZ = XY$
then $AB = XY$

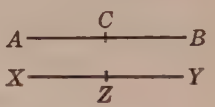


FIG. 37

Ax. 4. Halves of equals are equal, or, more generally, if equals are divided by equals, the results are equal.

If $\angle x = \angle y$ (Fig. 38)
and AE bisects $\angle x$
and BF bisects $\angle y$
then $\angle a = \angle b$

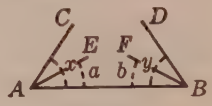


FIG. 38

Ex. 1. Illustrate Axioms 1 and 4 by using segments and Axioms 2 and 3 by using angles.

Ex. 2. Give algebraic illustrations of Axioms 3 and 4.

Ax. 5. Quantities equal to the same or to equal quantities are equal to each other.

1. If $\angle a = \angle x$
and $\angle b = \angle y$
and $\angle x = \angle y$
then $\angle a = \angle b$

2. If $AC = XY$
and $BD = XY$
then $AC = BD$

Ex. 3. Illustrate Axiom 5 by drawing two figures, one using angles and one using segments.

Ax. 6. Equals may be substituted for equals in statements of equality.

1. If $x = 4$
and $x + 5y = 29$
then $4 + 5y = 29$

2. If $\angle x = \angle a$
and $\angle x + \angle y = 180°$
then $\angle a + \angle y = 180°$

Fundamental Geometric Assumptions

52. Following is a second list of fundamental geometric assumptions or postulates to supplement those on page 19:

Post. 7. Any figure may be moved about in space without changing its size or shape.

Post. 8. Figures congruent to the same figure are congruent to each other.

Exercise. Bring into class illustrations of congruent figures (see note following §32, p. 20). Look for them among leaves, flower petals, figures on dress goods, or in advertisements. Illustrate Post. 8 with such figures.

Post. 9. An angle can be bisected by one and only one line.

Post. 10. One perpendicular and only one can be drawn to a line through a point in the line.

Post. 11. One perpendicular and only one can be drawn to a line from a point not in the line.

Post. 12. The perpendicular is the shortest segment from a point to a straight line.

Post. 13. All straight angles are equal.

Post. 14. All right angles are equal.

Post. 15.† Complements of the same or of equal angles are equal.

Post. 16. Supplements of the same or of equal angles are equal.

Post. 17. Vertical or opposite angles are equal.

Post. 18. If a ray starts from a point in a straight line, the sum of the two adjacent angles formed is a straight angle, or 180°.

†For informal proofs of Postulates 15–17 see A Exercises, page 34.

Post. 19. If any number of rays start from a point in a straight line and extend on the same side of that line, the sum of the adjacent angles formed is a straight angle, or 180°.

Post. 20. If two supplementary angles are adjacent, their exterior sides lie in a straight line.

C Exercises

Give the reasons required in the following exercises. These reasons are either some definition that you have learned, one of the axioms, §51, p. 29, or one of the postulates, §29, p. 19, and §52, p. 31. When you have found the correct reason, you have proved the statement.

For Fig. 39:

1. If $CB \perp BF$, why is $\angle CBF$ a right angle?
2. If $CB \perp AF$, why is $\angle 1$ the complement of $\angle 2$?
3. If AB is a straight line, why is $\angle AEC$ the supplement of $\angle 4$?
4. If CE and DB are straight lines, why is $\angle 11 = \angle 12$?
5. If $\angle 5 = 46°$, $\angle 6 = ?°$. Why?

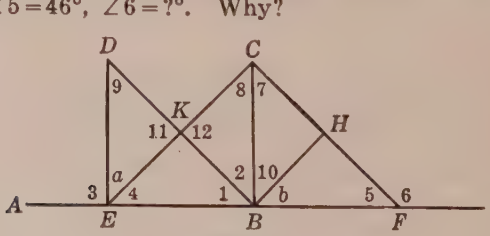


FIG. 39

6. If $\angle 4 = \angle 2$ and $\angle 2 = \angle 7$, why is $\angle 4 = \angle 7$?
7. If $DE \perp AB$, why is $\angle 3 = \angle DEB$?
8. If $\angle CBF = \angle CBE$, why is $CB \perp EF$?
9. If $DE \perp AB$ and $CB \perp AB$, why is $\angle 3 = \angle CBE$?
10. If $DE \perp AB$, $CB \perp AB$, and $\angle 4 = \angle 1$, why is $\angle 2 = \angle a$?

For Fig. 40:

11. If $AX = BY$ and $CX = CY$, why is $AC = BC$?

12. If WZ is a straight line and $\angle 3 = \angle 6$, why is $\angle DAO = \angle EBO$?

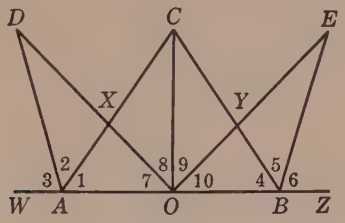


FIG. 40

13. If $\angle DAO = \angle EBO$ and $\angle 2 = \angle 5$, why is $\angle 1 = \angle 4$?

14. If $\angle 3 = 50°$ and $\angle 1 = 48°$, $\angle 2 = ?°$. Why?

15. If $\angle 7 = 42°$, and if $CO \perp AB$, $\angle 8 = ?°$ and $\angle DOB = ?°$. Why?

For Fig. 41:

16. If X is the mid-point of BC and Y the mid-point of AC and if $AC = BC$, why is $CX = CY$?

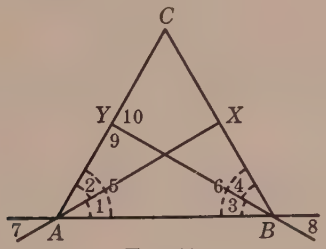


FIG. 41

17. If $\angle 5 = \angle 6$, and AX bisects $\angle 5$, and BY bisects $\angle 6$, why is $\angle 1 = \angle 3$?

18. If $AC = BC$ and $CY = CX$, why is $AY = BX$?

19. If $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$, why is $\angle 5 = \angle 6$?

20. If $\angle 7 = 25°$ and $\angle 5 = 60°$, $\angle 2 = ?°$. Why?

21. If $\angle 4 = 28°$ and $\angle 8 = 36°$, $\angle 6 = ?°$. Why?

For Fig. 42:

22. If $\angle a = \angle 4$, $\angle a = \angle 9$, and $\angle 4 = \angle 8$, why is $\angle 9 = \angle 8$?

23. If $\angle 4 = \angle 1$ and $\angle 2 = \angle a$, why is $\angle DEB = \angle CBE$?

24. If $\angle EBC = \angle ECF$ and $\angle 2 = \angle 8$, why is $\angle 1 = \angle 7$?

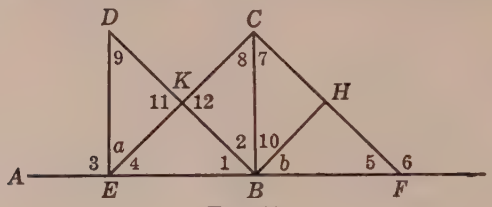


FIG. 42

25. If $CB \perp AF$ and $\angle 1 = \angle b$, why is $\angle 2 = \angle 10$?

26. If $\angle 1 = 45°$, $\angle b = 45°$, and $\angle 2 = 10°$, $\angle 2 = ?°$.

27. If $\angle 1 = 32°$ and $CB \perp EF$, $\angle 2 = ?°$. Why?

28. If $\angle 11 = 90°$, $\angle 12 = ?°$ Why?

29. If $\angle 4 = \angle 1$, why is $\angle AEC = \angle DBF$?

30. If $\angle 1 = b°$, and $\angle 2 = 10°$, why is $BC \perp EF$?

31. If $\angle 4 = \angle 5$, why is $\angle AEC = \angle 6$?

A Exercises

1. If $\angle x = \angle a$, $\angle y$ is the complement of $\angle x$, and $\angle b$ is the complement of $\angle a$, show that $\angle y = \angle b$ as below (Fig. 43):

$\angle x + \angle y = 1$ rt. $\angle$. Why?
$\angle a + \angle b = 1$ rt. $\angle$. Why?
$\therefore \angle x + \angle y = \angle a + \angle b$. Why?
$\therefore \angle y = \angle b$. Why?

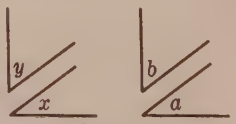


FIG. 43

The work outlined above proves that complements of equal angles are equal. See note at bottom of page 31.

2. Show by a method similar to that in Ex. 1 that supplements of equal angles are equal.

3. Show by Post. 16 that vertical angles are equal.

4. If, in Fig. 41, p. 33, the lines through point A are straight and AX bisects $\angle 5$, show that $\angle 7 = \angle 2$.

5. If, in Fig. 41, p. 33, $\angle 7 = \angle 8$ and the lines through points A and B are straight, show that $\angle 1 = \angle 3$.

6. If, in Fig. 41, p. 33, the lines through points A and B are straight, AX and BY bisect $\measuredangle 5$ and 6 respectively, and $\angle 5 = \angle 6$, show that $\angle 7 = \angle 8$.

SYMMETRY†

53. In many cases you can make one figure coincide with another, or part of a figure coincide with the remainder by folding it along some line as an axis, or by rotating it about some point as a center. Such figures are said to be **symmetric.**

A figure is said to be symmetric with respect to a line as an axis if one part will coincide with the remainder when it is folded on that line as an axis. Fig. 44 shows such a figure. AB is the axis.

FIG. 44

Two figures are said to be symmetric with respect to a line as an axis if one figure will coincide with the other when the plane in which they lie is folded on that line as an axis. Fig. 45 shows such figures. AB is the axis.

Such a figure or such figures are said to have axial symmetry.

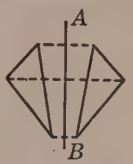


FIG. 45

Ex. 1. Find elsewhere in the book figures that have axial symmetry and name the axis in each case.

Ex. 2. Find the axis of symmetry of the end points of a segment.

Ex. 3. Has an angle an axis of symmetry? How would you find it?

†*To the teacher:* This outline of symmetry is for informal treatment only.

Courtesy of the artist, Samuel Yellin

FIG. 46. **Washington Memorial Chapel, Valley Forge, Pennsylvania.** These wrought-iron gates show forged, pierced, and carved work. They are considered one of the best representations of their type in the United States. Note the use of axial symmetry in the designs.

54. A figure is said to be symmetric with respect to a point as a center if one part of the figure will coincide with the remainder when it is rotated through an angle of 180° about the point as a center. In Fig. 47, O is the center of symmetry.

FIG. 47

Two figures are said to be symmetric with respect to a point as a center if one figure coincides with the other when it is rotated through an angle of 180° about the point as a center. In Fig. 48, O is the center of symmetry.

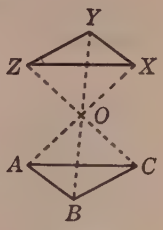


FIG. 48

Such a figure or such figures are said to have central symmetry.

Ex. 1. Find elsewhere in the book figures that have central symmetry and name the center of symmetry in each case.

Ex. 2. What letters of the alphabet have central symmetry? axial symmetry?

Ex. 3. Find the center of symmetry of the end points of a segment. Give the reason for your result.

55. Symmetric figures are much used in ornament. Designers make constant use of the idea of symmetry. Illustrations may be found in ornamental windows, in wall paper, and in nature, as in flowers and snow crystals. Engineers also use symmetry.

Ex. 1. Bring into class some drawing that you have made or some illustration, leaf, or flower that you have found that illustrates (*a*) axial symmetry; (*b*) central symmetry.

Ex. 2. Can you think of any figure which may have both axial and central symmetry?

Ex. 3. In the illustration on page 160 which figures have axial symmetry? Which have central symmetry? Which have both axial and central symmetry?

Ex. 4. Fig. 49 shows three symmetric encaustic tile designs. Tell the kind of symmetry in each.

FIG. 49

Ex. 5. How might kaleidoscopes and mirrors be used by designers?

Survey Test

1. (*a*) How many points are needed to read a straight line? (*b*) How many ends has a straight line? (*c*) How far may a straight line extend?

2. (*a*) Define a ray. (*b*) How many ends has a ray? (*c*) What is its origin?

3. (*a*) What is a segment? (*b*) By what other name is a segment known? (*c*) How many ends has it?

4. Draw a figure showing two segments that have been added. Write the statement indicating the addition.

5. How may a smaller segment be subtracted from a larger? Draw a figure illustrating such subtraction and write the statement indicating it.

6. When are two segments equal? congruent?

7. What is the segment joining two points often called?

8. Define an assumption; a postulate; an axiom.

9. (*a*) How many straight lines can be drawn through any one given point? through any two given points? through any three given points? (*b*) How many points are necessary to locate a straight line? (*c*) In how many points can two straight lines intersect?

10. (*a*) How many segments can be drawn between any two given points? (*b*) How many mid-points does a segment have?

11. How many rays can be drawn (*a*) from a given point? (*b*) from a given point and passing through a second given point?

12. (*a*) Define an angle. (*b*) What are its sides or arms? (*c*) What is its vertex?

13. What do you understand by congruent angles? by adjacent angles?

14. Name four ways in which an angle may be read. Illustrate by drawings.

15. Explain how to add two angles geometrically. Make a drawing to illustrate such addition and write the statement indicating it.

16. Explain how an angle may be obtained geometrically which is the difference between the smaller and the larger of two given angles. Make a drawing to illustrate such subtraction and write the statement indicating it.

17. Define a right angle in three ways.

18. Upon what does the size of an angle depend?

19. What do you understand by a straight angle? an acute angle? an obtuse angle?

20. Define complementary, supplementary, and vertical angles.

21. How can you find the supplement of an angle by the use of a protractor?

22. (*a*) What is the measure number of an angle? (*b*) When are angles said to be equal?

23. What can you say (*a*) of equal angles? (*b*) of congruent angles?

24. Find in the book figures of angles that illustrate each of the following:

(*a*) Equals added to equals give equals.

(*b*) Equals subtracted from equals give equals.

(*c*) Halves of equals are equal.

25. Complete the following:

Equals may be substituted for equals in .....

26. Show how the postulate quoted in answer to question 25 may be illustrated by the following:

If $\angle A = \angle B$ and $\angle B + \angle C = 2\text{rt. } \measuredangle$, $\therefore$ .....

27. (*a*) How many perpendiculars may be drawn to a line? Why? (*b*) Do Posts. 10 and 11, page 31, apply here? Why?

28. (*a*) Are all straight angles equal? Why? (*b*) Are all right angles equal? Why?

29. Quote the postulates that tell when angles are equal.

30. Make a drawing to illustrate complements of equal angles.

31. Make a drawing to illustrate supplements of equal angles.

32. (*a*) By what other name are vertical angles known? (*b*) Are vertical angles equal or unequal? Why?

33. (*a*) If a ray starts from a point in a straight line, how many degrees are there in the sum of the two adjacent angles formed? (*b*) What name is given to the two angles formed?

34. Draw three unequal angles whose sum is 180°. Quote the postulate that applies.

35. If two supplementary angles are adjacent, their exterior sides lie in what kind of line?

36. What do you understand by axial symmetry? by central symmetry?

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	16	13	20, 21	25	30
2	17	14	19, 20	27	31
3	17	15	21	28	31
4	17	16	21	29	31
5	18	17	22, 24	30	34
6	18	18	22	32	23, 31
7	18	19	22, 23	33	31, 23
8	19, 29	20	23	34	32
9	15, 19	21	24, 25	35	32
10	19	22	24	36	35, 37
11	19	23	24		
12	19	24	29, 30, 32, 33		

UNIT II

STRAIGHT-LINE FIGURES

56. Straight-line figures in the world about you. All sorts of straight-line figures are common in architecture, industry, and ornament. Glance over the diagrams in this unit and sketch the more common lines and figures that you are to study, such as parallels, perpendiculars, and the various kinds of triangles and quadrilaterals. What illustrations of these figures do you find in the world around you? Street car tracks, the boards in wood floors, and doors and windows all offer illustrations. Look at dress goods, purses, compacts, and jewelry for further examples. What figures do you think are most common? Cut out any such illustrations that you find and paste them on sheets of notebook paper. The pictures in the book will give you suggestions as to where to look for such figures. The advertising sections of magazines offer abundant material for such work.

TOPIC 1. CONGRUENT TRIANGLES I

57. Triangles in the world about you. Triangles may be more common than you think. In architecture you will find triangles used in interior rafter designs in churches and large assembly halls and made the basis for designs in many art-glass windows. Illustrations of such uses are found on pages 214 and 219. Can you tell which are the figures referred to?

In industrial ornament look for triangles in such things as tiles, linoleum, parquet floors, and iron grilles. See pages 107, 127, 139, and 321, and notice in how many ways triangles are used. The lattice pattern on page 105

is similar to many Arabic designs. The Arabs used these for window lattices and for wall decorations.

Exercise. Frames of various shapes can be made by fastening together straight pieces of wood or iron as shown in Fig. 50. Could you change the shape of any of these figures by pressing upon opposite sides or vertices? Which ones could be so changed?

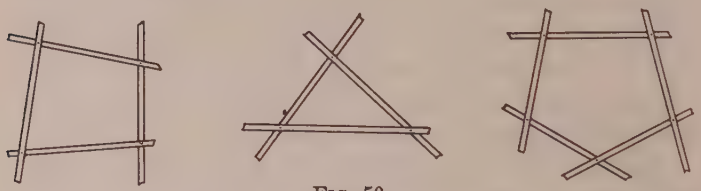
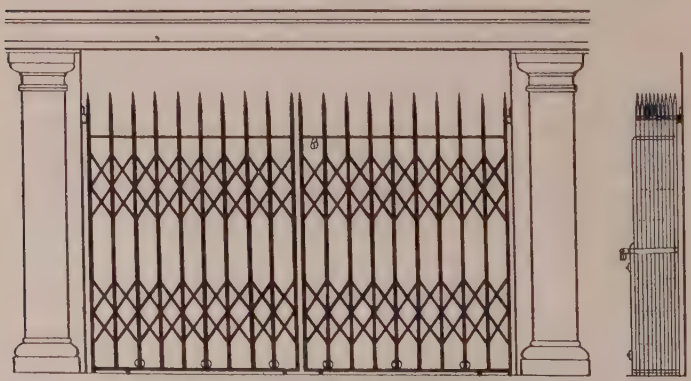


FIG. 50

Roof trusses, elevated train structures, bridges, derricks, fence gates, and the like are made rigid by dividing the spaces into triangles (see p. 43 and Fig. 132, p. 98). A roof truss is often in the form of an isosceles triangle (see Fig. 98, p. 69, and Fig. 256, p. 175).



From circular of William R. Pitt Composite Iron Work

FIG. 51. **A pair of folding gates.** The stiffening bars at the top and the track on which the gates roll at the bottom are hinged. When closed the gates are padlocked top and bottom.

Courtesy Chicago Bridge and Iron Works

FIG. 52. **Elevated tank with a capacity of 1,500,000 gallons.** This tank is owned by the city of Detroit, Michigan. Note the use of triangles to secure rigidity. Triangles give rigidity because they are the only polygons whose shape cannot be altered without a change in the length of their sides.

Exercise. If triangles in general are used for rigidity, how must the gates shown in Fig. 51 be made so that they can be folded back as shown at the right? Notice that the triangles do not show on the drawing of the closed gate.

58. A line made up of successive straight-line segments placed end to end so that they do not form a straight line is called a **broken line** (Fig. 53). Whenever the ends of a broken line meet, the line is called a **closed broken line** (Fig. 54).

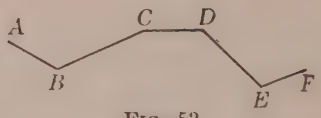


FIG. 53

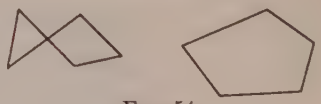


FIG. 54

A closed broken line is called a **polygon.**

59. A polygon of three sides is called a **triangle.** In Fig. 55, ABC is a triangle. It has three **sides,** AB, BC, and CA, and three **angles,** $\angle A$, $\angle B$, and $\angle C$. The points A, B, and C, which are the vertices of the angles, are also called the **vertices** of the triangle.

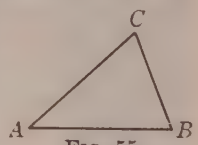


FIG. 55

60. A triangle with at least two sides equal is called an **isosceles triangle.** The equal sides are called the **legs.** Illustrate by a drawing.

61. A triangle with all of its sides equal is called an **equilateral triangle.** Illustrate by a drawing.

62. A triangle that contains a right angle is called a **right triangle.** The perpendicular sides are called the **legs** and the third side is called the **hypotenuse.** It is opposite the right angle. Illustrate by a drawing.

63. In any triangle ABC the side AB and the $\angle C$ are said to be opposite each other. Illustrate by a figure.

64. If one side of a triangle is horizontal, that side is often called the **base** of the triangle and the vertex opposite the base is called the **vertex** of the triangle. For many purposes it is necessary to consider any side as the base of the triangle. Then the angle opposite that side is the vertex of the triangle.

Exercise. Draw a triangle ABC. If AB is the base, which is the vertex? If AC is the base, which is the vertex? If BC is the base, which is the vertex?

65. In an isosceles triangle the angle made by the equal sides is called the **vertex angle** and its vertex is called the **vertex** of the triangle. The third side is called the **base** of the triangle.

66. The use of compasses. Compasses are used in drawing circles and parts of circles.

A closed curve every point of which is equally distant from a given point in the same plane is called a **circle**. The given point is called the **center** of the circle. A part of a circle between two points is called an **arc** (Fig. 56, arc AB). A segment from the center to the circle is called the **radius** (Fig. 56, CO). A segment through the center with both ends on the circle is called a **diameter** (Fig. 56, DE).

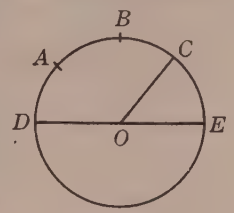


FIG. 56

In drawing circles and arcs, the point on which the sharp point of the compasses is placed is the center of the circle, and the distance between the points of the compasses is the radius of the circle.

67. You may assume as a postulate:

Post. 21. Radii of the same circle are equal.

FIG. 57

Ex. 1. Construct an isosceles triangle with base 3 cm. and the equal sides each 5 cm.

NOTE. In all exercises requiring the construction of figures, make the drawing and tell how you did it. Notice the wording for Ex. 1.

1. Make segment $AB = 3$ cm. (Fig. 57).

2. With 5 cm. as a radius and A as a center draw an arc above AB.

Complete the directions and the drawing.

Ex. 2. Construct (*a*) a triangle with sides 7 cm., 6 cm. and 5 cm.; (*b*) an equilateral triangle with each side 6 cm.

68. Congruence. Any two figures that can be made to coincide are called **congruent** ($\cong$) **figures.** In congruent triangles **corresponding sides or angles** are sides or angles that coincide or can be made to coincide. Notice that in congruent triangles corresponding sides are opposite angles that are known to be equal and corresponding angles are opposite sides that are known to be equal.

You are to study congruent triangles and their uses.

69. In any triangle ABC, the sides AC and BC are the sides of $\angle C$. The sides AC and BC include $\angle C$, and AC, $\angle C$, and BC are therefore spoken of as **two sides and the included angle.**

Ex. 1. Draw a triangle XYZ and name three cases of two sides and the included angle.

Ex. 2. Turn elsewhere in the book to figures that contain triangles. In each triangle select two sides and the included angle.

Ex. 3. Draw a triangle with two sides equal to the segments a and b (Fig. 58). Draw this triangle with a soft pencil on paper that is not too heavy. Compare your figure with your neighbor's by placing one paper upon the other and holding them to the light.

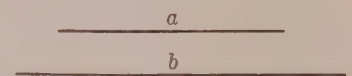


FIG. 58

Ex. 4. Draw a triangle with two sides equal to the segments a and b and the included angle equal to $\angle C$ (Fig. 59). Compare your figure with your neighbor's as explained in Ex. 3.

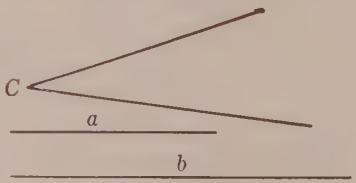


FIG. 59

70. A **theorem** is a statement of a fact that is to be proved true. What is stated about a figure is said to be **given.**

THEOREM 1*†

71. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent.

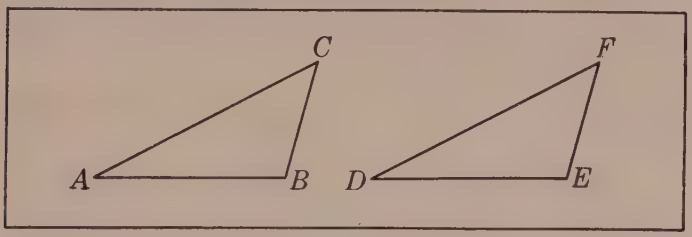


FIG. 60

Given $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $AB = DE$, and $AC = DF$.

To prove $\triangle ABC \cong \triangle DEF$.

Plan: The theorem is proved by superposition.

Proof:

STATEMENTS	REASONS
1. Place $\triangle ABC$ on $\triangle DEF$ so that AB will fall on DE, A on D, B on E, and C and F on the same side of DE.	1. This can be done because $AB = DE$. Quote Post. 7, p. 31.
2. Segment AC will fall along the line of DF.	2. $\angle A = \angle D$.
3. Point C will fall on point F.	3. $AC = DF$.
4. BC will coincide exactly with EF.	4. Quote Post. 4, p. 19.
5. $\triangle ABC \cong \triangle DEF$.	5. Two triangles that coincide are congruent.

*Theorems starred are those starred by the College Entrance Examination Board, and are the ones the candidate is expected to be able to prove upon examination.

† *To the teacher:* The formal proofs to Ths. 1 and 2 may be omitted or postponed.

72. The **meaning and use of Th. 1** may be explained by reference to figures elsewhere in the book: thus, in Fig. 65, p. 51, AB, $\angle 1$, and AX in $\triangle ABX$ and AC, $\angle 2$, and AX in $\triangle ACX$ are two sides and the included angle in each case. If, therefore, you can show that $AB = AC$, $\angle 1 = \angle 2$, and $AX = AX$, you know that the triangles can be made to coincide.

Exercise. Show how Th. 1 may be used in other figures on pp. 52–55.

73. In any triangle ABC, the side AB is between the vertex of $\angle A$ and the vertex of $\angle B$. $\angle A$, AB, and $\angle B$ are referred to as **two angles and the included side.**

Ex. 1. Draw a triangle XYZ and name three cases of two angles and the included side.

Ex. 2. Turn elsewhere in the book to figures that contain triangles. In each case name two angles and the included side.

Ex. 3. Draw a triangle with two angles equal to $\angle A$ and $\angle B$ (Fig. 61). Compare your triangle with your neighbor's by placing one paper upon the other and holding them to the light.

FIG. 61

Ex. 4. Draw a triangle with two angles equal to $\angle A$ and $\angle B$ and the included side equal to segment c (Fig. 62). Compare your triangle with your neighbor's by placing one paper upon the other and holding them to the light.

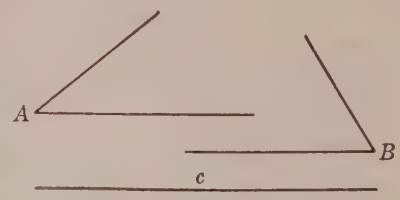


FIG. 62

THEOREM 2*

74. If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent.

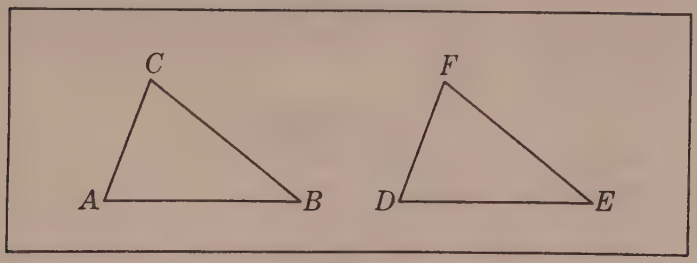


FIG. 63

Given $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$, and $AB = DE$.

To prove $\triangle ABC \cong \triangle DEF$.

Plan: The theorem is proved by superposition.

Proof:

STATEMENTS	REASONS
1. Place $\triangle ABC$ on $\triangle DEF$ so that AB falls on DE, A on D, B on E, and C and F on the same side of DE.	1. This can be done because . . . . . ?
2. AC will fall along the line of DF.	2. Why?
3. BC will fall along the line of EF.	3. Why?
4. C will fall on the line of DF and also on the line of EF.	4. Why?
5. $\therefore$ C will fall on point F.	5. Quote Post. 3, p. 19.
6. $\therefore$ $\triangle ABC \cong \triangle DEF$.	6. Why?

75. Use of congruent triangles. Ths. 1 and 2 are to be used to prove segments and angles equal. It is necessary first to be able to prove two triangles congruent.

Ex. 1. Review §72, p. 48.

Ex. 2. Explain the meaning and use of Th. 2.

Ex. 3. Show how Th. 2 might be used by referring to figures on pp. 51–55.

C Oral Exercises

Use Fig. 64 for the following exercises:

1. Name the angles included by each of the following pairs of sides: AE and ED; AD and DC; EB and BD; EA and AC.

2. Name the side included by each of the following pairs of angles: $\angle 9$ and $\angle 5$; $\angle 3$ and $\angle 1$; $\angle 10$ and $\angle 8$; $\angle 6$ and $\angle 8$.

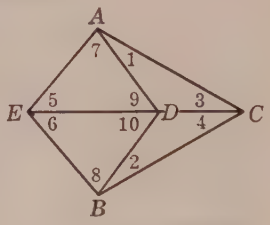


FIG. 64

By what theorem would $\triangle ADC$ and $\triangle BDC$ be proved congruent if:

3. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and $AC = BC$?

4. $AC = BC$, $\angle 3 = \angle 4$, and $CD = CD$?

5. $AC = BC$, $AD = BD$, and $\angle 1 = \angle 2$?

6. $\angle 3 = \angle 4$, $\angle ADC = \angle BDC$, and $CD = CD$?

7. $\angle ADC = \angle BDC$, $\angle 1 = \angle 2$, and $AD = BD$?

Complete the following sentences:

$\triangle ADE$ and $\triangle BDE$ would be congruent by

8. Th. 1, if $AE = BE$, $DE = DE$, and . . . . . .

9. Th. 2, if $\angle 5 = \angle 6$, $\angle 9 = \angle 10$, and . . . . . .

10. Th. 1, if $AE = BE$, $\angle 7 = \angle 8$, and . . . . . .

11. Th. 2, if $\angle 5 = \angle 6$, $AE = BE$, and . . . . . .

$\triangle AEC$ and $\triangle BEC$ would be congruent by

12. Th. 1, if $AE = BE$, $\angle 5 = \angle 6$, and . . . . . .

13. Th. 2, if $AE = BE$, $\angle 5 = \angle 6$, and . . . . . .

C Exercises

Be sure to notice what is given about the figures in the following exercises. It is what you have to "go on" in your solution of the exercises, and must be used.

1. Exercise 1 is worked out. Complete the proof by answering the questions.

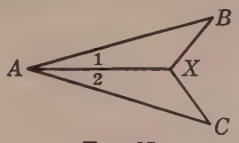


FIG. 65

Given AX bisects $\angle A$ and $AB = AC$ (Fig. 65).

Prove $\triangle ABX \cong \triangle ACX$.

Proof:

	STATEMENTS	REASONS
1.	$\angle 1 = \angle 2$.	1. Why?
2.	$AB = AC$.	2. Why?
3.	$AX = AX$.	3. Common side.
4.	$\therefore \triangle ABX \cong \triangle ACX$.	4. Why?

Statements 1, 2, and 3 above give the equal parts that make the triangles congruent. These three statements determine the reason for statement 4. Where do the reasons for statements 1, 2, and 3 come from?

Notice that in using Ths. 1 and 2 you do not place one triangle upon the other each time. You merely show that this can be done and tell whether it can be done by the method of Th. 1 or that of Th. 2.

All figures should be drawn accurately from the data given.

2. Given (Fig. 65) $\angle 1 = \angle 2$ and $\angle AXB = \angle AXC$. Prove $\triangle AXB \cong \triangle AXC$.

Use Ex. 1 above as a model, for both method and arrangement in the exercises that follow.

3. Given (Fig. 66) $CA = CD$ and CX bisects $\angle C$. Prove $\triangle ACX \cong \triangle DCX$.

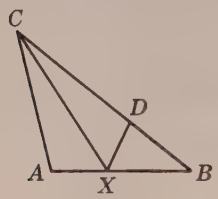


FIG. 66

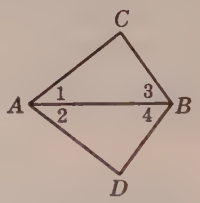


FIG. 67

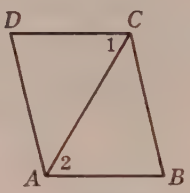


FIG. 68

4. Given (Fig. 67) AB bisects both $\angle A$ and $\angle B$.
Prove $\triangle ABC \cong \triangle ABD$.

5. Given (Fig. 67) AB bisects $\angle A$ and $AC = AD$.
Prove $\triangle ABC \cong \triangle ABD$.

6. Given (Fig. 68) $\angle 1 = \angle 2$ and $DC = AB$.
Prove $\triangle ABC \cong \triangle ACD$.

7. In each of the foregoing Exs. 2–6 name the equal parts that would prove the triangles congruent.

76. To prove segments and angles equal. If, in Ex. 1, page 51, $\triangle ABX$ is proved congruent to $\triangle ACX$, they could be made to coincide. In that case BX would coincide with and be equal to CX. You would say $BX = CX$, **corresponding sides of congruent triangles are equal.** Similarly $\angle B$ would be equal to $\angle C$, **corresponding angles of congruent triangles are equal.**

Ex. 1. In Ex. 4 above, if $\triangle ABC$ is proved congruent to $\triangle ABD$, which sides are equal because they are corresponding parts of these triangles? Which angles are equal for the same reason?

Ex. 2. If in Ex. 2, p. 51, $\triangle AXB$ is proved congruent to $\triangle AXC$, name one pair of corresponding sides that would have to be equal. Give the reason.

Ex. 3. In Exs. 5 and 6 above, how could "corresponding parts of congruent triangles are equal" be used in each case?

77. One-step analyses. In connection with each exercise and theorem, there will be given a brief statement of the plan to be used in solving that exercise or theorem. This plan is called an **analysis.** In the following exercises the analysis will consist of one sentence. This sentence will name the triangles to be used to prove the required segments or angles equal; thus in Ex. 1 below:

To prove $DO = CO$, *prove* $\triangle AOD \cong \triangle BOC$.

A sentence of this type should be so written that it will occupy only one line and should be given for each exercise.

Give the analysis for 2–8 following and for B Exercises 1–6.

C Exercises

1. Notice that this exercise is worked out. Complete the proof required by answering the questions.

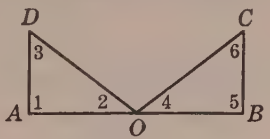


FIG. 69

Given (Fig. 69) DA and $CB \perp AB$, $DA = CB$, and O the mid-point of AB.

Prove $DO = CO$.

Analysis: To prove $DO = CO$, prove $\triangle AOD \cong \triangle BOC$.

Proof:

	STATEMENTS	REASONS
1. *a.*	$AO = BO$.	1. *a.* Why?
b.	$\angle 1 = \angle 5$.	*b.* Why?
c.	$DA = CB$.	*c.* Given.
d.	$\therefore \triangle AOD \cong \triangle BOC$.	*d.* Why?
2.	$\therefore DO = CO$.	2. Why?

Use Ex. 1 as a model for both method and arrangement of work for the exercises below.†

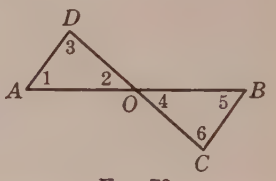


FIG. 70

2. Given (Fig. 70) O the mid-point of AB and $\angle 1 = \angle 5$. Prove $\angle 3 = \angle 6$ (see p. 31, Post. 17).

3. Given (Fig. 70) AB bisecting CD and CD bisecting AB. Prove $AD = BC$.

4. Given (Fig. 70) O the mid-point of DC. $AD \perp DC$ and $BC \perp DC$. Prove $\angle 1 = \angle 5$.

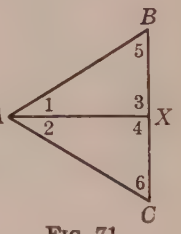


FIG. 71

5. Given (Fig. 71) AX bisecting $\angle A$ and $BC \perp AX$. Prove $AB = AC$ and $BX = XC$.

6. Given (Fig. 71) $AX \perp BC$ and AX bisecting BC. Prove $AB = AC$.

7. Given (Fig. 72) AC and BD intersecting at O, $\angle 1 = \angle 2$, $AO = BO$. Prove $DO = CO$.

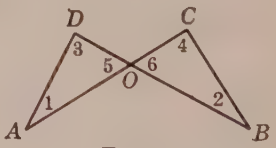


FIG. 72

8. Given (Fig. 72) AC and BD intersecting at O, $DO = CO$, $AD \perp DO$, and $BC \perp CO$. Prove $\angle 1 = \angle 2$.

B Exercises

1. Given (Fig. 73) $AB = DC$, $AE = DF$, and $\angle 3 = \angle 4$. Prove $EC = FB$ (use Ax. 1, p. 29).

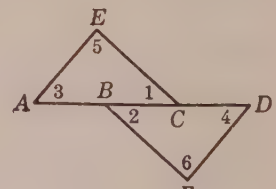


FIG. 73

2. Given (Fig. 73) $AB = DC$, $\angle A = \angle D$, and $\angle 1 = \angle 2$. Prove $AE = DF$ and $EC = FB$.

3. Given (Fig. 74) BO bisects $\angle B$ and $BA = BC$. Prove $AO = CO$.

† *To the teacher:* You will find a review of C Exercises, pages 32–34, profitable at this point.

4. Given (Fig. 74) BO bisects $\angle B$, $\angle 7 = \angle 8$. Prove $BA = BC$ (see Post. 16, p. 31).

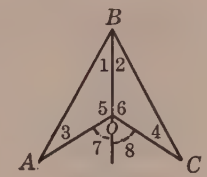


FIG. 74

5. Given (Fig. 75) $AX = BY$, ZX and WY both $\perp AB$, and $\angle 1 = \angle 6$.
Prove $ZX = WY$.

6. Given (Fig. 75) $AX = BY$, ZX and WY both $\perp AB$, and $ZX = WY$.
Prove $\angle 1 = \angle 6$.

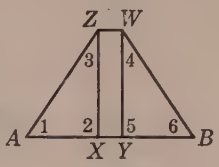


FIG. 75

7. Construct an isosceles triangle ABC, with base AB. Let CO, the bisector of $\angle C$, meet the base at O. Prove that the segments joining O with the mid-points of AC and BC are equal.

8. Show that congruent triangles may be used to find the distance across a pond as follows (Fig. 76):

First set up a stake at any convenient point, such as O, from which points A and B are both visible. Set up a stake at D in a straight line with O and A so that $OD = OA$. In the same way place stake C so that $OC = OB$. Measure CD.

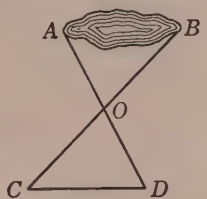


FIG. 76

9. Show that the distance across a stream may be measured as follows (Fig. 77): A pole is set up at A with a stick fastened at the top so that $\angle ACD$ may be altered by moving the stick CD. A person stands with the eye at C and moves the stick CD until he just sees point B. The pole is then turned around. Standing with eye at C and looking along CD', he locates point B'. He then measures AB'. Could you do this using the visor of your cap rather than a pole and a stick? Is it necessary to turn through 180°? This device is said to be a very old one.

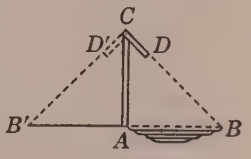


FIG. 77

10. Explain from Fig. 78 how to find the distance across a stream. Notice that the required distance AB is perpendicular to a line along the bank of the stream.

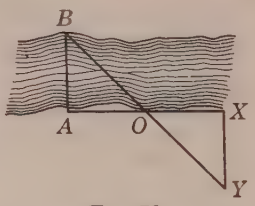


FIG. 78

NOTE. Theorems 1 and 2 were probably known to Thales. Thales is regarded as the founder of one of the earliest Greek schools of mathematics, about 600 B.C. It is said that he used Th. 2 to determine the distance of a ship at sea. It is not known how this was done. He might have used a tower or cliff and a method similar to that of Ex. 9, p. 55.

78. Construction lines. Sometimes the proof of an exercise or of a theorem is difficult or even impossible without the use of some line or lines that are not given. Such helping lines that are not given are called **construction lines.** A definite statement of how such lines are located or drawn should always be made before the proof begins. Such a line is used in Th. 3, p. 57.

79. I. **In studying Th. 3** first memorize the theorem.

II. Be sure that you know what is given and what is to be proved. How were these facts obtained?

III. How should Fig. 79, No. 1, be drawn?

IV. How have you proved two angles equal before this? See p. 54, Ex. 2 of C Exercises, and compare with Th. 3, analysis, statement 1.

V. Look at Fig. 79, No. 1 and No. 2. What is the purpose of line CO? How is it drawn? Read statements 2 and 3 of the analysis. (See Ex. 15, p. 26.)

VI. Think out the reasons called for in the proof.

VII. Close your book and try to write out all the work, beginning with the statement of the theorem, using different letters.

THEOREM 3*

80. The angles opposite the equal sides of an isosceles triangle are equal.

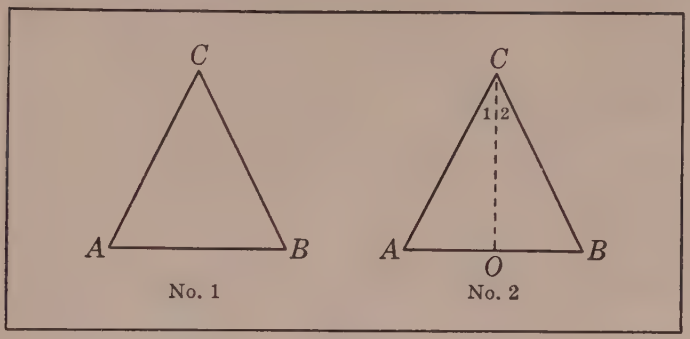


FIG. 79

Given the isosceles $\triangle ABC$ with $AC = BC$.

To prove $\angle A = \angle B$.

Analysis and construction:

1. To prove $\angle A = \angle B$, prove $\angle A$ and $\angle B$ corresponding angles of congruent triangles.
2. To obtain the triangles, bisect $\angle C$ and continue the bisector CO to meet the base at O.
3. $\therefore$ to prove $\angle A = \angle B$, prove $\triangle AOC \cong \triangle BOC$.

Proof:	STATEMENTS	REASONS
1. *a.*	$AC = BC$.	1. *a.* Why?
b.	$\angle 1 = \angle 2$.	*b.* Why?
c.	$CO = CO$.	*c.* Why?
d.	$\therefore \triangle AOC \cong \triangle BOC$.	*d.* Why?
2.	$\therefore \angle A = \angle B$.	2. Why?

Exercise. Turn elsewhere in the book and find figures showing isosceles triangles and name their base angles.

Theorem 4*

81. If three sides of one triangle are equal to three sides of another triangle, the triangles are congruent.

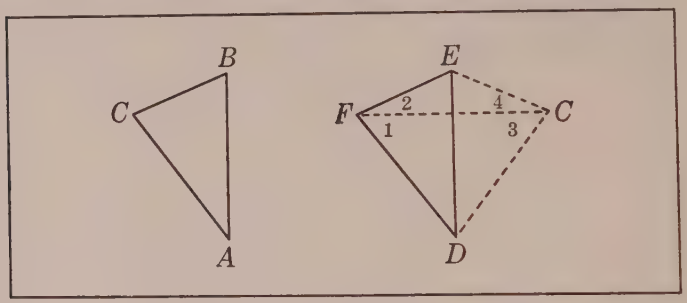


Fig. 80

Given $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $AC = DF$, and $CB = FE$.

To prove $\triangle ABC \cong \triangle DEF$.

Analysis and construction:

1. To prove $\triangle ABC \cong \triangle DEF$, prove $\angle C = \angle F$.
2. To prove $\angle C = \angle F$, place $\triangle ABC$ so that AB, the longest side of $\triangle ABC$, coincides with DE, the longest side of $\triangle DEF$, and point C is opposite point F. Draw CF and prove that $\angle C$ and $\angle F$ are made by adding the base angles of two isosceles triangles.

Proof:

STATEMENTS	REASONS
1. *a.* $EF = EC$.	1. *a.* Const. and Ax. 5, p. 30.
b. $\triangle EFC$ is isosceles.	*b.* A triangle that has two sides equal is isosceles.
c. $\angle 2 = \angle 4$.	*c.* The base angles of an isosceles $\triangle$ are equal.

2. *a.*	$DF = DC$.	2. *a.* Why?
b.	$\triangle DFC$ is isosceles.	*b.* Why?
c.	$\angle 1 = \angle 3$.	*c.* Why?
3.	$\therefore \angle F = \angle C$.	3. Quote Ax. 1, p. 29.
4.	$\triangle DEF \cong \triangle DEC$.	4. Why?
5.	$\triangle DEF \cong \triangle ABC$.	5. Quote Post. 8, p. 31.

82. I. **In studying Th. 4,** first memorize the theorem.

II. Be sure that you understand what is given and what is to be proved.

III. Draw ABC any triangle; then draw $\triangle DEF$ from what is given.

IV. What theorems have been used to prove two triangles congruent? How could one of these be made to apply? Which theorem would be the most convenient to use? Why? See statement 1 of the analysis. The entire analysis is probably too hard for you to work out by yourself. Study the text carefully. What is the construction? What is its purpose? Close your book and try to write it all out.

Exercises†

1. What two facts concerning an isosceles triangle can be used in a proof when a triangle is given isosceles?

2. Quote three theorems that can be used to prove two triangles congruent.

3. Prove Th. 4 by proving $\angle B = \angle FED$; by proving $\angle A = \angle FDE$.

4. (*Honor*) Prove Th. 4, p. 58, by proving $\angle A = \angle FDE$ when $\angle C$ and $\angle F$ are obtuse angles.

† *To the teacher:* It may not be necessary to give all of these exercises to any one class.

5. Given (Fig. 81) $\triangle ABC$ isosceles, $AX = BY$, and WX and $ZY \perp AB$ at X and Y respectively. Prove $WX = ZY$.

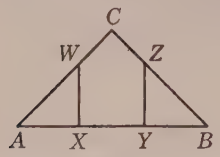


FIG. 81

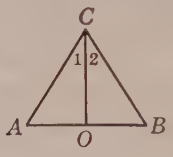


FIG. 82

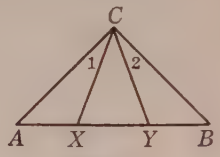


FIG. 83

6. Given (Fig. 81) $\triangle ABC$ isosceles, $AX = BY$, and $AW = BZ$. Prove $XW = YZ$.

7. Given (Fig. 82) $\triangle ABC$ isosceles and O the mid-point of AB. Prove $\angle 1 = \angle 2$.

8. Given (Fig. 83) $\triangle ABC$ isosceles and $AX = BY$. Prove $CX = CY$.

9. Given (Fig. 83) $\triangle ABC$ isosceles and $\angle 1 = \angle 2$. Prove $CX = CY$.

10. Given (Fig. 84) $\triangle ABC$ isosceles, O the mid-point of AB, and $AX = BY$. Prove $OX = OY$.

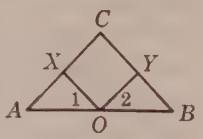


FIG. 84

11. Given (Fig. 84) $\triangle ABC$ isosceles, O the mid-point of AB, and $\angle 1 = \angle 2$. Prove $AX = BY$.

12. Given (Fig. 85) $\triangle ABC$ isosceles and CD bisecting $\angle C$. Join D, any point in CD, with A and B. Prove $AD = BD$.

13. Given (Fig. 85) $\triangle ABC$ and $\triangle ABD$ isosceles. Prove $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

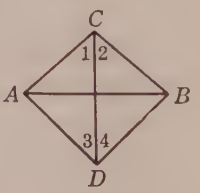


FIG. 85

83. In studying problems like that in §84, you should be prepared to make the drawing and to give the directions, analysis, and proof.

Problem 1

84. At a given point in a given line to construct an angle equal to a given angle.

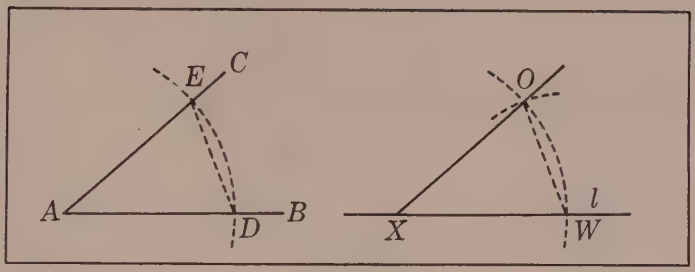


Fig. 86

Given line l, point X in line l, and $\angle BAC$.

To construct on line l at point X an angle equal to $\angle BAC$.

Directions:

1. With any radius and A as a center draw an arc cutting the sides of $\angle A$ at D and E.
2. With the same radius and X as a center draw an arc cutting line l at W.
3. With DE as a radius and W as a center draw an arc cutting the last arc at O.
4. Join X and O. $\angle WXO$ is the required angle.

Analysis: To prove $\angle X = \angle A$, draw DE and WO and prove $\triangle XWO \cong \triangle ADE$.

Proof: STATEMENTS	REASONS
1. *a.* $XO = AE$.	Give all reasons. The reasons for 1, *a*, *b*, and *c*, are found in the directions.
b. $XW = AD$.	
c. $WO = DE$.	
d. $\therefore \triangle XWO \cong \triangle ADE$.	
2. $\therefore \angle A = \angle X$.	

Problem 2

85. To construct the bisector of a given angle.

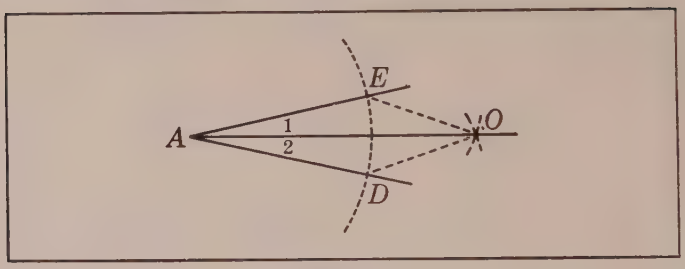


Fig. 87

Given $\angle A$.

To construct the bisector of $\angle A$.

Directions:

1. With any convenient radius and A as a center draw an arc cutting the sides of $\angle A$ at D and E.
2. With any convenient radius and E as a center draw an arc within the angle.
3. With the same radius and D as a center cut the last arc at O.
4. Join A and O. AO is the required bisector.

Analysis:

1. To prove AO bisects $\angle A$, prove $\angle 1 = \angle 2$.
2. " " $\angle 1 = \angle 2$, { draw EO and DO, prove $\triangle AEO \cong \triangle ADO$.

Proof:

	STATEMENTS	REASONS
1. *a.*	$AE = AD$.	Give all reasons.
b.	$EO = DO$.	
c.	$AO = AO$.	
d.	$\therefore \triangle AEO \cong \triangle ADO$.	
2.	$\therefore \angle 1 = \angle 2$.	
3.	$\therefore AO$ bisects $\angle A$.	

86. Two-step analyses. Where have you had analyses of only one step? Where have you had analyses of more than one step? When analyses are too hard for you to make without help, the book will give you the help needed. In other cases you can make your own analyses.

To make an analysis, it is necessary to ask yourself a series of questions. Start with what is to be proved and by means of these questions unravel the proof step by step.

To make the analysis for the proof to Th. 3, you should begin by asking yourself, How can I prove two angles equal? The answer to this question gives the first step of the analysis (see Th. 3, p. 57) and suggests the construction given in step 2.

To make the analysis for the proof of Th. 4, you should begin by asking yourself, What theorems will prove two triangles congruent? The answer to this question is, Either Th. 1 or Th. 2. If you notice carefully what is given, it is evident that Th. 1 may be used if you can prove that one angle of one triangle is equal to the corresponding angle of the other (see Th. 4, p. 58).

To make the analysis for the proof to Prob. 2, p. 62, you should begin by asking yourself, How can I prove that an angle is bisected? The answer is, By proving that it is divided into two equal parts. This answer gives step 1 in the analysis (see p. 62). At this point a second question is needed, namely: How can I prove two angles equal? The answer to this question gives step 2 of the analysis.

Notice in the analysis to Prob. 2 that the last part of step 1 is the first part of step 2. This is often true in a good analysis. Whenever possible, write the analysis so that each step occupies only one line and use ditto marks. The steps of the proof then stand out. The proof is the analysis done backward.

Problem 3

87. To construct a perpendicular to a line at a given point in the line.

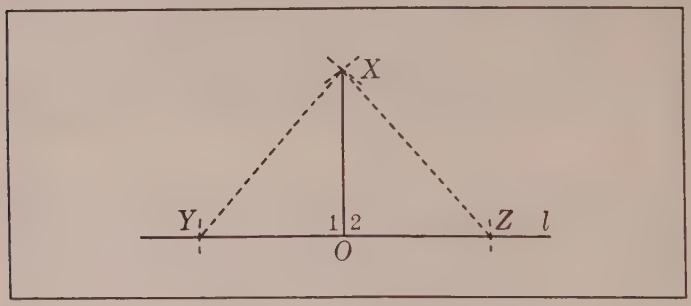


Fig. 88

Given line l and point O in line l.

To construct a perpendicular to line l at point O.

Directions:

1. With any convenient radius and O as a center draw arcs cutting line l at Y and Z.
2. With a longer radius and Y as a center draw an arc.
3. With the same radius and Z as a center cut the last arc at X.
4. Join X and O. OX is the required perpendicular.

Analysis:

1. To prove $XO \perp$ line l, prove $\angle 1 = \angle 2$.
2. " " $\angle 1 = \angle 2$, $\begin{cases} \text{draw } XY \text{ and } XZ, \\ \text{prove } \triangle XOY \cong \triangle XOZ. \end{cases}$

Work the analysis backward to get the proof. What is the object (1) in drawing XY and XZ? (2) in proving $\triangle XOY \cong \triangle XOZ$? (3) in proving $\angle 1 = \angle 2$? Show that in Prob. 3 you are bisecting a straight angle.

NOTE. **Trisection of angles.** By the method given in §85 you can bisect any angle you choose. The trisection of an angle is a much more difficult problem. You will find a little later how to trisect a right angle. All work in elementary geometry is confined to the study of the circle and straight line, and no instruments are used except the compasses and the straightedge. A straightedge is a ruler that is not graduated to any scale. It has been shown that angles generally cannot be trisected by the use of these instruments only. The Greeks learned how to trisect any angle, and since their time many ways of trisecting angles have been found. These methods, however, have always required curves other than the circle and instruments other than the compasses and straightedge.

Draftsmen's methods for trisecting angles are approximations.

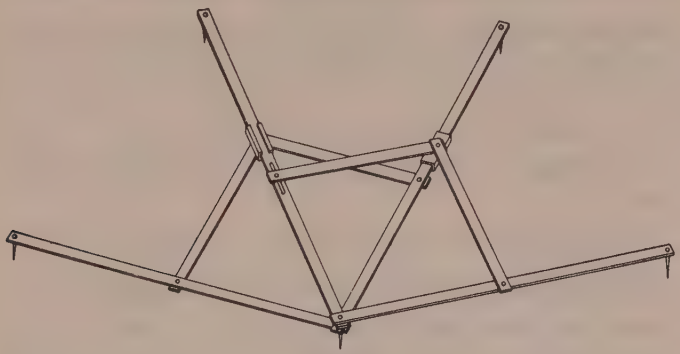


FIG. 89

A number of different instruments for trisecting angles have been made. Fig. 89 shows one such instrument. What segments are made equal? Show that Fig. 87, p. 62, is used twice in Fig. 89. Why is the angle trisected? How could an instrument for bisecting angles be made? Such an instrument is sometimes used by carpenters for cutting and fitting moldings.

Exercises for Ruler and Compasses†

1. Construct a triangle equal to a given triangle. In how many ways may this be done?

†*To the teacher:* It may not be necessary to give all of these exercises to any one class.

2. From the data given, construct the figures for the following exercises: p. 51, Ex. 1; p. 53, Ex. 1; p. 54, Exs. 2 and 5 of C Exercises; p. 54, Ex. 1 of B Exercises; p. 55, Exs. 4 and 5; p. 60, Exs. 5, 9, and 11.

3. Construct an angle of 45°, 135°, 22½°, 157½°.

4. Construct two supplementary adjacent angles. Construct the bisector of each. How many degrees are there in the angle made by the bisectors? Why?

5. Construct the complement of an acute angle. Answer question 4 for the two complementary adjacent angles formed.†

C Exercises

Two-step analysis should be used in the exercises below and on page 67, as follows: When it is required to prove that

(1) An angle is bisected (as in Prob. 2, p. 62).

(2) One line is perpendicular to another (as in Prob. 3, p. 64).

(3) A triangle is isosceles (as in Ex. 1 below).

(4) A segment is bisected (as in Ex. 2 following).

Draw the figure from the data given in the exercise. Write the analysis with each sentence on a new line and use ditto marks. Review §24, p. 17, and §25, p. 18, §§34 and 35, p. 21; also Posts. 15–17, p. 31.

1. In Fig. 90, $\triangle ABC$ is isosceles and CO bisects $\angle C$.
Prove $\triangle ABX$ isosceles.

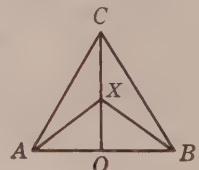


FIG. 90

Analysis:

I. To prove $\triangle ABX$ isosceles, prove $AX = BX$.
II. " " $AX = BX$, " . . . . .

† *To the teacher:* At this point some of the exercises on pages 27-28 may be worked with ruler and compasses if you desire.

2. Given $\triangle ABC$ isosceles (Fig. 91) and CO bisecting $\angle C$. Prove CO bisects AB.

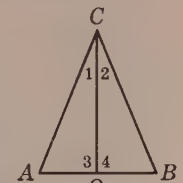


FIG. 91

Analysis:

I. To prove CO bisects AB, prove $AO = BO$.

II. " " $AO = BO$, prove $\triangle AOC \cong \triangle BOC$.

3. Using the data given in Ex. 2, prove $CO \perp AB$.

4. Given $\triangle ABC$ isosceles (Fig. 91) and CO bisecting AB. Prove that CO bisects $\angle C$.

5. Using the data given in Ex. 4, prove $CO \perp AB$.

B Exercises

1. Given (Fig. 92) XA and $YB \perp AB$, $XA = YB$. Prove $YA = XB$.

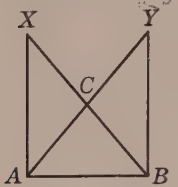


FIG. 92

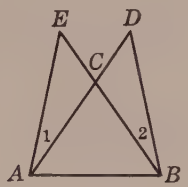


FIG. 93

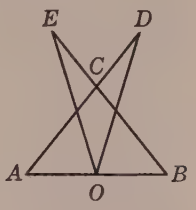


FIG. 94

2. Given (Fig. 92) $\triangle ABC$ isosceles, XA and $YB \perp AB$ at A and B respectively. Sides AC and BC are extended to meet the $\perp$s at Y and X respectively. Prove $XA = YB$.

3. Given (Fig. 93) $\triangle ABC$ isosceles. BC and AC are extended so that $CE = CD$. Draw EA and DB. Prove $EA = DB$.

4. Given (Fig. 93) $\triangle ABC$ isosceles and $\angle 1 = \angle 2$. AC and BC are extended to meet the sides of $\angle 1$ and $\angle 2$ at D and E respectively. Prove $AE = BD$.

5. Given (Fig. 94) $\triangle ABC$ isosceles, O the mid-point of AB, and $CD = CE$. Prove $OE = OD$.

6. Given (Fig. 95) $\triangle ABC$ isosceles, O the mid-point of AB, and $AD = BE$. Prove $OD = OE$.

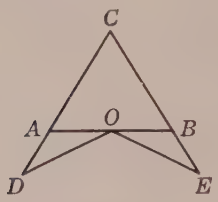


FIG. 95

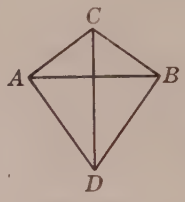


FIG. 96

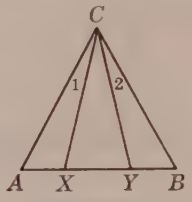


FIG. 97

7. In Fig. 95, omit OD and OE. Given $\triangle ABC$ isosceles, $AD = BE$. Draw BD and AE. Prove $BD = AE$.

NOTE. The figures for Exs. 6 and 7 are obtained from the outlines of steel trusses.

8. Given (Fig. 96) $\triangle ABC$ isosceles. Draw CD bisecting $\angle C$. Take D any point in the bisector extended. Draw AD and BD. Prove that CD bisects $\angle D$.

9. Prove Ex. 8 if D and C are on the same side of AB.

10. Given (Fig. 97) $\triangle ABC$ isosceles and $\angle 1 = \angle 2$. Prove $\triangle XYC$ isosceles.

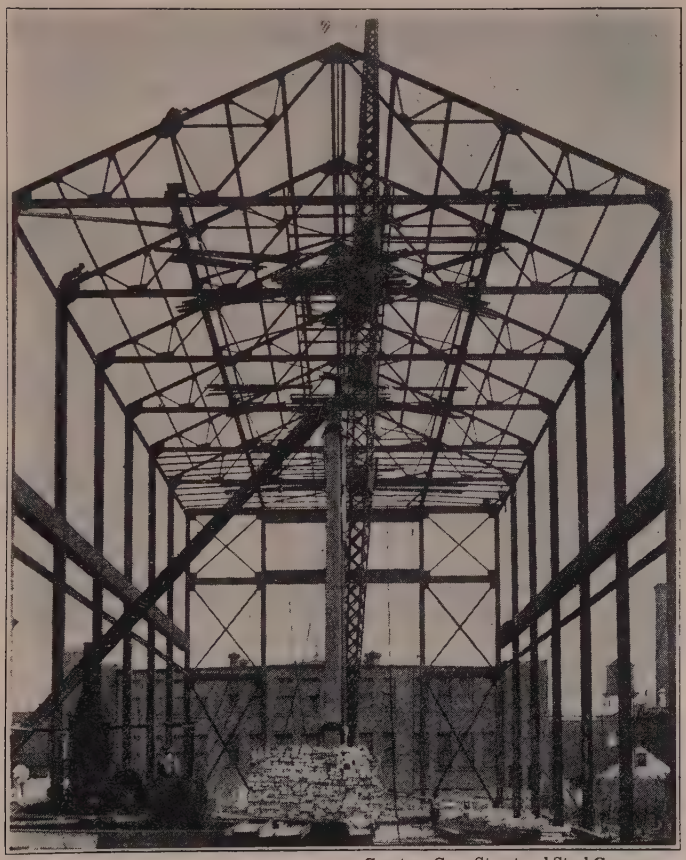
11. Given (Fig. 97) $\triangle ABC$ isosceles and $AX = BY$. Prove $\triangle XYC$ isosceles.

12. Given (Fig. 97) $\triangle XYC$ isosceles and $\angle 1 = \angle 2$. Prove $\triangle ABC$ isosceles.

13. Given (Fig. 97) $\triangle XYC$ isosceles and $XA = YB$. Prove $\triangle ABC$ isosceles.

14. Draw an isosceles $\triangle ABC$, with CA and CB the equal sides. Let X and Y be the mid-points of CA and CB respectively. Draw BX and AY and prove $BX = AY$.

15. Draw an isosceles $\triangle ABC$, with CA and CB the equal sides. Draw AX bisecting $\angle A$ and meeting CB at X. Draw BY bisecting $\angle B$ and meeting CA at Y. Prove $AX = BY$.



Courtesy Gage Structural Steel Company

FIG. 98. **Roof trusses on a building in the course of erection.** Roof trusses are nearly always in the shape of a triangle or a combination of triangles. A triangle does not change its shape under pressure. Triangles therefore give rigidity to a structure such as the one illustrated. For other forms of trusses see page 68, Exs. 6 and 7; page 77, Ex. 10; and page 175, Fig. 256.

88. Nature of theorems and proofs. The statements of geometrical facts receive different names: definitions, assumptions, theorems, and corollaries.

A **theorem**, as you learned on page 46, is a statement of a fact that is to be proved true.

An **assumption** is a statement of a fact accepted as true without proof. Assumptions, as you have already learned, are also called axioms and postulates. Many of the assumptions have been obtained by observation. Many have been shown to be true informally.

The proof of a theorem shows that if certain facts are known to be true a certain other fact *must* be true. The facts that are **known** or **given** constitute the **hypothesis.** The fact to be **proved** is called the **conclusion.**

In Theorem 3 the hypothesis is: The triangle has two equal sides. The conclusion is: The angles opposite these sides are equal.

In Theorem 4 the hypothesis is: Three sides of one triangle are equal to three sides of another triangle. The conclusion is: The triangles are congruent.

The hypothesis and conclusion of a theorem may be found by studying the grammatical construction of the statement of the theorem. If the sentence contains a clause beginning with "if," this clause is the hypothesis. If there is no such clause, the complete subject is the hypothesis and the complete predicate the conclusion.

In each case the proof consists in showing that the conclusion *must* follow from the hypothesis. The analysis shows how the proof has been or may be thought out. The proof is the analysis worked backward and is set down in what is called the synthetic form. The synthetic method is the opposite of the analytical method.

A theorem that follows easily from another theorem is called a **corollary** of that theorem.

Proofs are required for all the theorems, corollaries, and exercises that follow unless some specific statement is made to the contrary. In all proofs a reason must be given for each step.

Some of the exercises have been called problems. Any **geometrical problem** calls for the construction of some geometrical figure to fulfill certain stated requirements. Unless it is otherwise stated, these constructions must be performed with compasses and straightedge only. In all problems it is necessary to prove that the completed figure fulfills the stated requirements.

C Exercises

When theorems or postulates are referred to or stated in the following exercises, draw the figure, letter it, and give the hypothesis and conclusion in terms of the letters of your figure.

1. Postulates 13–20, pages 31 and 32.

2. Restate Postulates 13–17 so that each contains a clause beginning with "if."

3. If two equilateral triangles have a side of one equal to a side of the other, the triangles are congruent.

4. If two right triangles have a leg and an adjacent acute angle of one equal respectively to a leg and an adjacent acute angle of the other, the triangles are congruent.

5. If two isosceles triangles have a leg and the vertex angle of one equal respectively to a leg and the vertex angle of the other, the triangles are congruent.

6. Restate Exs. 3, 4, and 5 so that they do not contain a clause beginning with "if."

7. Make statements for the following exercises that shall not contain any letters: page 60, Exs. 7 and 13; page 68, Exs. 14 and 15.

Draw the figures for Exs. 8, 9, and 10. Give the hypothesis and conclusion in terms of the letters of your figure. The proofs for Ex. 8 and Ex. 9 are given on page 67.

8. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

9. The segment which joins the vertex of an isosceles triangle with the mid-point of the base bisects the vertex angle and is perpendicular to the base.

10. Any point in the bisector of the vertex angle of an isosceles triangle is equally distant from the ends of the base.

89. Summary of important methods used.

I. If you are to prove two segments equal, look for:
   a. Sums, differences, equal multiples, or equal parts of equal segments.
   b. Sides of isosceles or of equilateral triangles.
   c. Corresponding sides of congruent triangles.

II. If you are to prove two angles equal, look for:
   a. Sums, differences, equal multiples, or equal parts of equal angles.
   b. Right angles.
   c. Supplements of equal angles.
   d. Complements of equal angles.
   e. Vertical angles.
   f. Base angles of an isosceles triangle.
   g. Corresponding angles of congruent triangles.

III. To prove two triangles congruent, use:
   a. Theorem 1, two sides and the included angle.
   b. Theorem 2, two angles and the included side.
   c. Theorem 4, three sides.

THEOREM 5†

90. If two points are each equally distant from the ends of a segment, the line passing through these points is the perpendicular bisector of the segment.

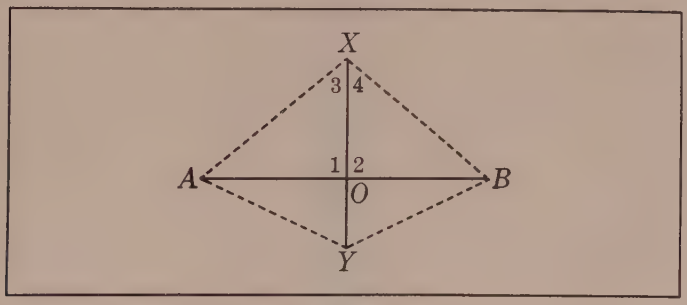


FIG. 99

Hypothesis: AB is any segment, and points X and Y are so located that $AX = BX$ and $AY = BY$.

Conclusion: $XY \perp AB$ and XY bisects AB.

Analysis:

1. To prove $XY \perp AB$, prove $\angle 1 = \angle 2$.
2. " " XY bisects AB, " $AO = BO$.
3. " " $\left\{\begin{matrix}\angle 1 = \angle 2 \\ AO = BO\end{matrix}\right\}$, " $\triangle AOX \cong \triangle BOX$.
4. " " $\triangle AOX \cong \triangle BOX$, " $\angle 3 = \angle 4$.
5. " " $\angle 3 = \angle 4$, " $\triangle AXY \cong BXY$.

Notice carefully the analysis above. What is the object in proving each pair of triangles congruent? Is the second pair really necessary? How do you get the proof from the analysis? Work out the proof in full, putting in all steps and reasons.

† *To the teacher:* The study of pages 73–78 may be postponed until beginning Topic 4, page 118, if you desire.

Problem 4

91. To construct a perpendicular to a line from a given point not in the line.

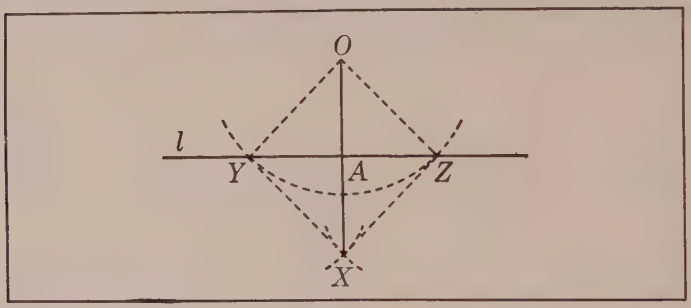


Fig. 100

Given line l and point O not in line l.

To construct a perpendicular to line l from point O.

Directions:

1. With any radius and O as a center draw an arc cutting line l at Y and Z.
2. With a radius greater than half YZ and Y as a center draw an arc opposite O.
3. With the same radius and Z as a center cut this arc at X.
4. Join O and X. OX is the required perpendicular.

Analysis: To prove $OX \perp l$, draw OY, OZ, XY, and XZ and prove $OY = OZ$ and $XY = XZ$.

Proof: STATEMENTS	REASONS
1. $OY = OZ$.	1. Why?
2. $XY = XZ$.	2. Why?
3. $\therefore OX \perp YZ$.	3. Quote Th. 5, p. 73.
4. $\therefore OX \perp l$.	4. Why?

Problem 5

92. To construct the perpendicular bisector of a given segment.

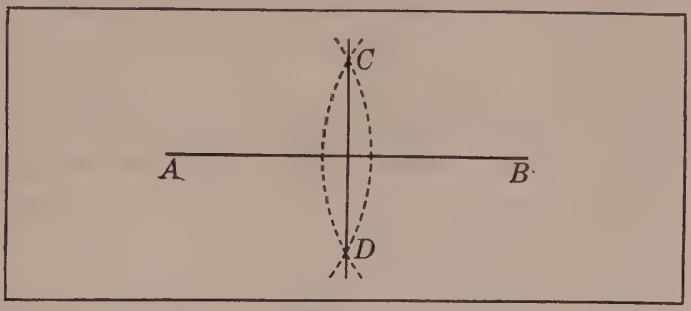


Fig. 101

Given the segment AB.

To construct the perpendicular bisector of AB.

Directions:

1. With any radius greater than $\frac{1}{2}AB$ and A as a center construct arcs above and below AB.
2. With the same radius and B as a center cut these arcs at C and D.
3. Join C and D.

For analysis and proof, see Prob. 4.

Ex. 1. Solve Prob. 4 with O and X on the same side of line l.

Ex. 2. Why is it desirable in solving Prob. 4 to have points O and X on opposite sides of line l?

Ex. 3. In the solution for Prob. 5 which radii must be made equal if Th. 5, p. 73, is to be used in the proof? Draw the figure in as many ways as possible, but be sure that Th. 5 will apply. Which is the most convenient way to draw the figure?

93. A segment joining any vertex of any triangle with the midpoint of the opposite side is called a **median** of the triangle. Therefore any triangle has three medians. In Fig. 103, CO is the median to side AB of triangle ABC.

B Exercises

Many of the following exercises require more than one pair of congruent triangles for the proof, and many have analyses of more than two steps. Think carefully.

1. The segments joining the mid-points of the sides of an isosceles triangle form an isosceles triangle.

2. In Fig. 102, $AC = AD$, $BC = BD$, and $AX = AY$. Prove $BX = BY$.

3. Prove Ex. 2 if X and Y are on AC and AD extended.

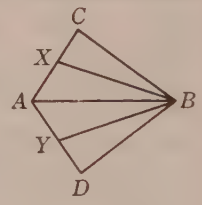


FIG. 102

4. If from the ends of the base of an isosceles triangle segments are drawn making equal angles with the base and terminated by the opposite sides, these segments are equal.

5. Any point in the median to the base of an isosceles triangle is equally distant from the ends of the base. Are there any special positions of this point for which your proof will not hold?

6. In Fig. 103, $\triangle ABC$ is isosceles. CO is the median to the base AB. If $CX = CY$, prove that CO bisects $\angle YOX$.

7. Corresponding medians of congruent triangles are equal.

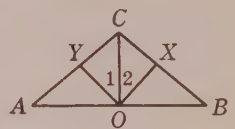


FIG. 103

8. In Fig. 103, $\triangle ABC$ is isosceles. CO is the median to the base AB. If $\angle 1 = \angle 2$, prove $CX = CY$.

9. Prove Ex. 8 if X and Y are on BC and AC extended.

10. Fig. 104 shows a common form of truss. The king-rod CO joins C with the mid-point of AB in the isosceles $\triangle ABC$. The braces OE and OF join O with E and F, the mid-points of AC and BC respectively. Prove that $OE = OF$ and that CO is perpendicular to AB.

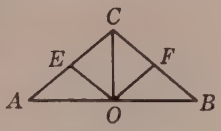


FIG. 104

11. In Fig. 105, CD is any segment. $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Prove that CD is a perpendicular bisector of AB.

12. If, in Fig. 105, $\triangle ABC$ is constructed isosceles and D, any point in the bisector of $\angle C$, is joined to A and B, DA and DB make equal angles with AB.

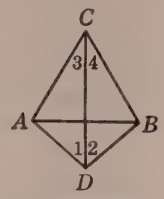


FIG. 105

13. Construct the figures for Exs. 11 and 12 with C and D on the same side of AB and give proofs.

14. Show how the instrument shown in Fig. 106 may be used as a leveling device. The pieces are so fastened together that $AB = AC$. AD is a plumb line. How must it be placed if BC is level? This instrument is said to be very ancient.

Suggestion. Notice that a level line is one that is perpendicular to a plumb line.

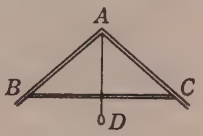


FIG. 106

15. In Fig. 107, $CA = CB$ and $AD = BD$. If X is any point in DC extended, prove that $AX = BX$.

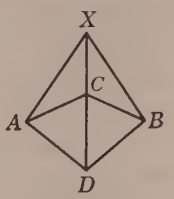


FIG. 107

16. Show that a carpenter's steel square may be used to bisect an angle as follows: Mark off equal distances OA and OB on the sides of the angle. Place the square as shown in Fig. 108. Mark point D. Join D and O.

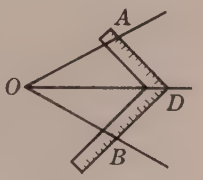


FIG. 108

A Exercises

1. Prove that the base angles of an isosceles triangle are equal by means of the construction shown in Fig. 109. $\triangle ABC$ is the isosceles $\triangle$. $CX = CY$.

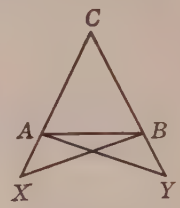


FIG. 109

Analysis:

To prove $\angle CAB = \angle CBA$, prove that $\angle YAC = \angle XBC$ and that $\angle YAB = \angle XBA$.

NOTE. This method of proving the theorem is given in Euclid's *Elements*. In the thirteenth century the students of Oxford, England, nicknamed this theorem "elefuga"—that is, "the flight of the wretched"—because most of them found it so difficult that few cared to study the subject further. Two or three hundred years later the students called it "pons asinorum," or "the bridge of asses." Very little is known concerning Euclid himself. He was a Greek who lived and taught in Alexandria about 300 B.C. It is said that Ptolemy once asked Euclid "if there was in geometry any shorter way than that of the *Elements*," and he answered that "there was no royal road to geometry." Another story is that someone who had begun to read geometry with Euclid asked, when he had learned the first theorem, "But what shall I get by learning these things?" Euclid called his slave and said, "Give him threepence, since he must make gain out of what he learns."

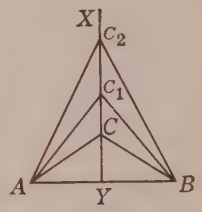


FIG. 110

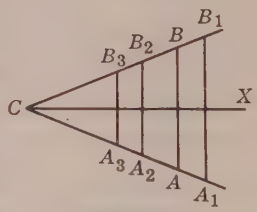


FIG. 111

2. In Fig. 110, XY is the perpendicular bisector of AB. If point C moves along the ray YX, how will the sides of $\triangle ACB$ change? What are the upper and lower limits of these sides? No proof is required.

3. In Fig. 111, CX bisects $\angle ACB$. If the segment AB moves to the right and to the left along the ray CX but is always perpendicular to CX, how will the sides of $\triangle BCA$ change? What is the lower limit of the length of these sides? What is the upper limit? No proof is needed.

Survey Test

1. What is a broken line? If its end points meet, what is it called?

2. (*a*) What is a polygon? (*b*) Why is a triangle a polygon? (*c*) Name its parts.

3. (*a*) Define isosceles, equilateral, and right triangles. (*b*) Why may an equilateral triangle be considered isosceles?

4. Which sides of an isosceles triangle are called the legs?

5. Which side of a right triangle is called the hypotenuse? What are the other two sides called?

6. (*a*) What do you understand by the base of a triangle? by the vertex? (*b*) Which side is considered the base in an isosceles triangle? (*c*) In an isosceles triangle which angle is considered the vertex angle?

7. For what purposes are compasses used in geometry?

8. Explain the difference between a circle and an arc.

9. (*a*) Explain the difference between a diameter and a radius. (*b*) Is there any limit to the number of radii a circle may have? (*c*) What can you say of the radii of the same circle?

10. What are congruent figures?

11. Draw a triangle. Letter it *DEF*. Number the angles 1, 2, and 3. Name two sides and the included angle. Name two angles and the included side.

12. What do you understand by corresponding sides and angles of congruent triangles?

13. (*a*) Define a theorem. (*b*) What is meant by *given?*

14. Quote three theorems by which two triangles may be proved congruent.

15. What special method have you in this chapter for proving segments and angles equal?

16. What kinds of exercises have you had that have one-step analyses? two-step analyses?

17. (*a*) In general, how do you make an analysis? (*b*) How do you get the proof of a theorem from the analysis?

18. What purpose do construction lines serve?

19. What two facts do you know about an isosceles triangle?

20. Draw an angle. Construct at a given point in a given line an angle equal to the given angle.

21. Construct the bisector of a given angle.

22. Construct a perpendicular to a line at a given point in the line.

23. Can angles in general be trisected by the use of a straightedge and compasses only?

24. How can you find the hypothesis of any theorem or exercise? How can you find the conclusion?

25. How is the perpendicular bisector of a segment determined?

26. Construct a perpendicular to a line from a given point not in the line. How many such perpendiculars can be constructed from the given point?

27. Construct the perpendicular bisector of a given segment.

28. Define the median of a triangle. How many medians does any triangle have?

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	43, 44	11	46, 48	20	61
2	44	12	46	21	62
3	44	13	46	22	64
4	44	14	47, 49, 58	23	65
5	44	15	52	24	70
6	44, 45	16	53, 66	25	73
7	45	17	63	26	74, 31
8	45	18	56	27	75
9	45	19	44, 57	28	76
10	46				

TOPIC 2. PARALLELS AND PERPENDICULARS

94. The practical use of parallel and perpendicular lines. Do you know parallel lines and perpendicular lines when you see them? They are commonly used on various game boards, on the football field and tennis court. A simple illustration of the importance of parallel lines is a railroad track. What would happen if the rails were not the same distance apart all along the line? Where do parallels and perpendiculars appear in window and door frames? You will learn later other ways in which such lines are used, as in gates (page 98) and architectural and construction work (page 115).

95. Definite location of lines. What is meant by construction lines? (See §78, page 56.) It is not always necessary that such lines be located definitely. When they are located definitely, care should be taken that no facts are assumed that require proof. The postulates that contain the phrase "only one line" are used to locate lines definitely. In general, a line is located by two points or by one point and a direction.

Ex. 1. Quote all postulates that can be used to locate a line, segment, or ray definitely (see p. 19 and pp. 31–32).

Ex. 2. Which of the postulates quoted in answer to Ex. 1 locate the line by one point and a direction? Give in each case the words in the postulate which determine the direction.

Ex. 3. Which of the following directions locate the required line or ray definitely? Why?

a. Construct a perpendicular to line AB.
b. Construct a perpendicular to line AB through point O.
c. From point O outside AB draw a ray cutting AB.
d. Draw a perpendicular bisector of segment AB.
e. Draw a segment bisecting segment AB.
f. Draw a ray that shall make a given angle with a given line.

96. Transversals and angles. A straight line that crosses two or more other straight lines is called a **transversal.** By the intersection of a transversal with two straight lines, various angles are formed, as follows:

$\angle c$, $\angle d$, $\angle w$, and $\angle x$ are **interior** angles (Fig. 112).

$\angle a$, $\angle b$, $\angle y$, and $\angle z$ are **exterior** angles.

$\angle c$ and $\angle x$, also $\angle d$ and $\angle w$, are **alternate interior** angles.

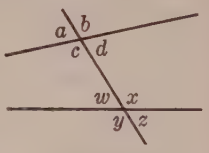


FIG. 112

$\angle a$ and $\angle z$, also $\angle b$ and $\angle y$, are **alternate exterior angles.**

$\angle a$ and $\angle w$, $\angle b$ and $\angle x$, $\angle c$ and $\angle y$, also $\angle d$ and $\angle z$, are **corresponding angles.**

Exercise. Find illustrations of the kinds of angles named above on p. 91, Fig. 123; p. 94, Fig. 128; p. 99, Figs. 133, 134; p. 100, Fig. 136; p. 101, Fig. 142; p. 103, Fig. 145.

97. Exterior angles of triangles. If one side of a triangle is extended, an angle is formed which is called an **exterior angle** of the triangle. In Fig. 113, $\angle 1$ is an exterior angle. Why? $\angle 3$ and $\angle 4$ are called the interior angles **non-adjacent** to $\angle 1$. Why?

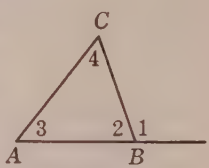


FIG. 113

Ex. 1. How many exterior angles in different positions has a triangle?

Ex. 2. If no two angles of a triangle are equal, how many exterior angles of different sizes has the triangle?

98. The following new axioms will be needed:

Ax. 7. *The whole is greater than any of its parts.*

Ax. 8. *If the first quantity is greater than the second and the second is greater than or equal to the third, then the first is greater than the third.*

THEOREM 6

99. An exterior angle of a triangle is greater than either of the non-adjacent interior angles.

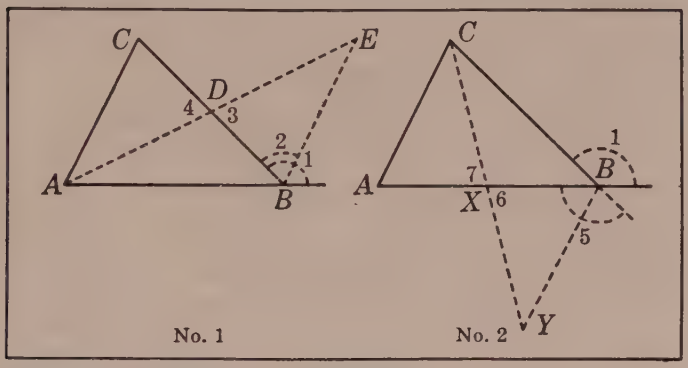


FIG. 114

Hypothesis: $\triangle ABC$ with the exterior $\angle 1$.

Conclusion: $\angle 1 > \angle C$ or $\angle BAC$.

Analysis and construction (Fig. 114, No. 1):

A. 1. To prove $\angle 1 > \angle C$, prove part of $\angle 1 = \angle C$.

2. " " part of $\angle 1 = \angle C$, bisect CB at D, draw AD, and extend, making $DE = AD$. Draw EB and prove $\angle 2 = \angle C$.

3. To prove $\angle 2 = \angle C$, prove $\triangle BDE \cong \triangle ADC$.

Proof:

STATEMENTS	REASONS
1. $\triangle BDE \cong \triangle ADC$. $\therefore \angle 2 = \angle C$.	1. Give all details.
2. $\angle 1 > \angle 2$.	2. Ax. 7, p. 82.
3. $\therefore \angle 1 > \angle C$.	3. Ax. 8, p. 82.

Analysis and construction:

B. 1. To prove $\angle 1 > \angle BAC$, extend CB and prove $\angle 5 = \angle 1$ and $\angle 5 > \angle BAC$ (Fig. 114, No. 2).

Give the construction and proof for B.

100. Lines drawn in the same plane will either intersect or they will not intersect. Lines in the same plane that do not intersect no matter how far they are extended are called **parallel lines.**

Exercise. Bring into class two or more pictures not in this book showing the use of parallel lines.

You are to study parallel lines and are to learn:

(1) Methods of proving lines parallel (Ths. 7–11).

(2) Important facts about parallels (Ths. 12–15).

Methods of proving lines parallel must of necessity be based upon the definition of parallel lines. See proofs of Ths. 7 and 8. Other methods will be learned later.

101. *Post. 22. Only one line can be drawn through a given point parallel to a given line, or two lines that intersect cannot both be parallel to the same line.*

102. The proof of Th. 7, p. 85, is an **indirect proof.** It is very different from any proof that you have met so far. In studying it, notice:

A. There are two possibilities in regard to lines a and b: either (1) $a \parallel b$, or (2) a is not $\parallel b$. Either one or the other of these possibilities must be true, but they cannot both be true at the same time. The proof consists in eliminating the second.

B. The method by which the second possibility is eliminated should be carefully noted. Study statements 3 and 4 of the proof. They contradict each other. Whenever, in a course of reasoning, you obtain a conclusion that is impossible, something is wrong somewhere. What was wrong in the proof of Th. 7? Proofs of this kind are called proofs by **reductio ad absurdum.** Why? See whether you can translate the Latin.

THEOREM 7

103. Two lines parallel to a third line are parallel to each other.

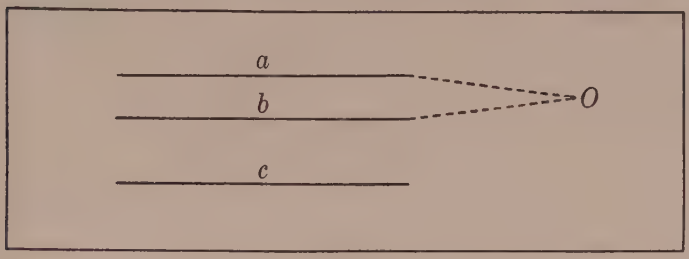


FIG. 115

Hypothesis: Line $a \parallel$ line c and line $b \parallel$ line c.

Conclusion: Line $a \parallel$ line b.

Analysis: To prove $a \parallel b$, prove that a and b cannot meet.

Proof:

STATEMENTS	REASONS
1. Either $a \parallel b$ or a is not $\parallel b$.	
2. If a were not $\parallel b$, then a and b would meet at some point as at O.	2. Quote the definition of parallel lines.
3. Then you would have a and b	3.
(1) both passing through O;	(1) Supposition.
(2) both $\parallel c$.	(2) Given.
4. But it is impossible to have two lines through the same point both $\parallel$ to the same line.	4. Post. 22.
5. $\therefore a$ and b cannot meet.	5. See §102, B.
6. $\therefore a \parallel b$.	6. Why?

104. Proofs like that for Th. 7 are sometimes called **proofs by elimination.** Why?

THEOREM 8

105. If two straight lines are cut by a third straight line so that the alternate interior angles are equal, the lines are parallel.

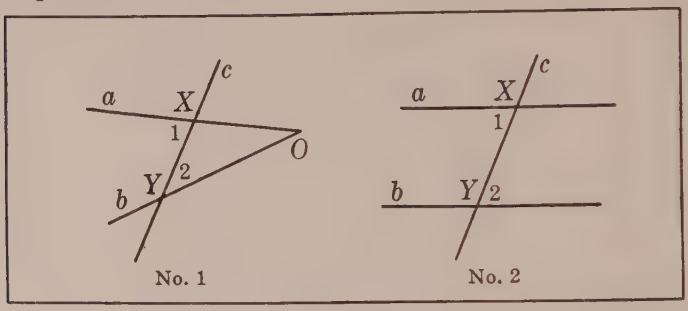


FIG. 116

Hypothesis: Lines a and b are cut by line c. $\angle 1 = \angle 2$.

Conclusion: Line $a \parallel$ line b.

Analysis: To prove $a \parallel b$, prove that a and b cannot meet.

Proof:

STATEMENTS	REASONS
1. Either $a \parallel b$ or a is not $\parallel b$.	
2. If a were not $\parallel b$, then a and b would meet at some point as O.	2. Why?
3. If a and b met, then $\angle 1$ would not equal $\angle 2$.	3. Th. 6, p. 83.
4. But $\angle 1 = \angle 2$.	4. Given.
5. $\therefore$ a and b cannot meet.	5. Why?
6. $\therefore$ $a \parallel b$.	6. Why?

Ex. 1. If a met b on the right, which would be greater, $\angle 1$ or $\angle 2$? If a met b on the left, which would be the greater? Why?

Ex. 2. Explain fully how the proof of Th. 8 fulfills the requirements for an indirect proof given in §102, page 84.

THEOREM 9

106. If two straight lines are cut by a third straight line so that one pair of corresponding angles are equal, the lines are parallel.

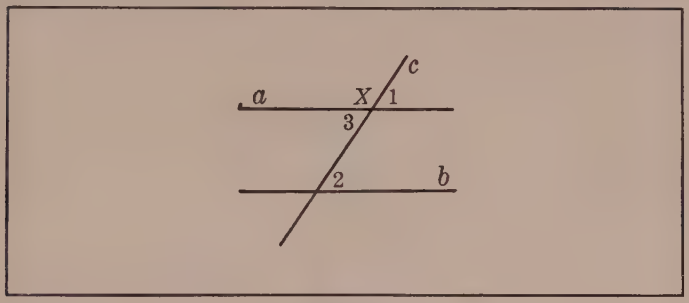


FIG. 117

Hypothesis: Lines a and b are cut by line c, and $\angle 1 = \angle 2$.

Conclusion: Line $a \parallel$ line b.

Analysis: To prove $a \parallel b$, prove $\angle 2 = \angle 3$.

Proof:

STATEMENTS	REASONS
1. $\angle 1 = \angle 2$.	1. Why?
2. $\angle 1 = \angle 3$.	2. Why?
3. $\therefore \angle 2 = \angle 3$.	3. Ax. 5, p. 30.
4. $\therefore a \parallel b$.	4. Th. 8.

107. Th. 9 might have been proved by an indirect proof. It is rather cumbersome to prove that two lines cannot meet every time you wish to prove two lines parallel. Why was it necessary to prove Ths. 7 and 8 by this method? Note the analysis for Ths. 9, 10, and 11.

Ex. 1. Construct Fig. 116, No. 2, as follows: Draw line b and the transversal c. At X any point in c other than Y construct $\angle 1 = \angle 2$. Line a should come parallel to b. Why?

Ex. 2. Construct Fig. 117, making $\angle 1 = \angle 2$.

Theorem 10

108. If two straight lines are cut by a third straight line so that the interior angles on the same side of the transversal are supplements, the lines are parallel.

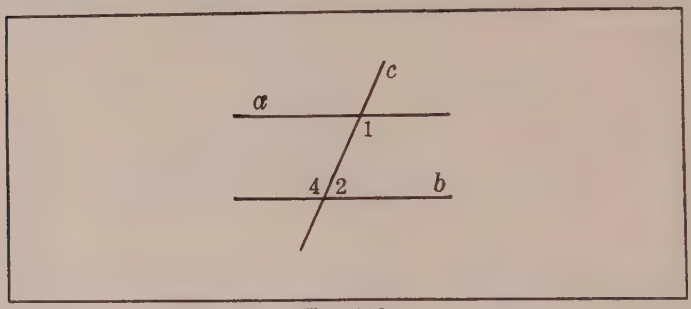


Fig. 118

Hypothesis: Lines a and b are cut by line c so that $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.

Conclusion: Line $a \parallel$ line b.

Analysis: To prove $a \parallel b$, prove $\angle 1 = \angle 4$.

Proof:

STATEMENTS	REASONS
1. $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.	1. Why?
2. $\angle 4 + \angle 2 = 2$ rt. $\measuredangle$.	2. Why?
3. $\therefore \angle 1 = \angle 4$.	3. Post. 16, p. 31.
4. $\therefore a \parallel b$.	4. Th. 8, p. 86.

Show whether a would be parallel to b in each of Exs. 1–7 below. If $a \parallel b$, give the reason in full (Fig. 119).

Ex. 1. $\angle 4 = 100°$ and $\angle 5 = 100°$.
Ex. 2. $\angle 1 = 110°$ and $\angle 8 = 110°$.
Ex. 3. $\angle 2 = 55°$ and $\angle 5 = 110°$.
Ex. 4. $\angle 7 = 60°$ and $\angle 4 = 120°$.
Ex. 5. $\angle 2 = 80°$ and $\angle 7 = 95°$.
Ex. 6. $\angle 1 = 115°$ and $\angle 7 = 75°$.
Ex. 7. $\angle 8 = 90°$ and $\angle 2 = 90°$.

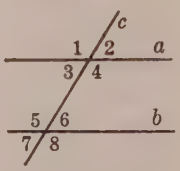


Fig. 119

THEOREM 11

109. Two straight lines in the same plane perpendicular to the same straight line are parallel.

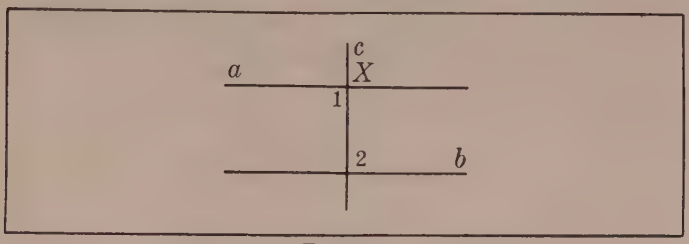


FIG. 120

Hypothesis: Lines a and b are $\perp$ line c.

Conclusion: Line $a \parallel$ line b.

Analysis: To prove $a \parallel b$, prove $\angle 1 = \angle 2$.

Outline of proof. Give the reasons.

1. $a \perp c$.
2. $\therefore \angle 1 =$ a rt. $\angle$.
3. $b \perp c$.
4. $\therefore \angle 2 =$ a rt. $\angle$.
5. $\therefore \angle 1 = \angle 2$.
6. $\therefore a \parallel b$.

110. COR.* **Only one perpendicular can be drawn to a line from a point outside the line.**

Use an indirect proof. See Th. 11, and Post. 22, p. 84.

111. Sometimes there are more than two possibilities in indirect proofs. For **indirect proofs in general:**

A. Determine all possible cases obtained by contradicting the given conclusion.

B. Then, since either the conclusion or one of the contradictory statements must be true, you must eliminate all but one of these by proving them absurd.

Problem 6

112. To draw a straight line through a given point parallel to a given straight line.

Three methods are given. In each case below, line a is to be constructed through X parallel to b:

Method 1. Fig. 116, No. 2, p. 86. Construct $\angle 1 = \angle 2$. Prove by Th. 8.

Method 2. Fig. 117, p. 87. Construct $\angle 1 = \angle 2$. Prove by Th. 9.

Method 3. Fig. 120, p. 89. Construct $c \perp b$ from X and $a \perp c$ at X. Prove by Th. 11. (*Optional.*)

Ex. 1. Draw any $\angle BAC$, not a right angle. From any point in side AB construct a line parallel to AC. From any point in AC construct a line parallel to AB.

Ex. 2. Make the construction called for in Ex. 1, making $\angle BAC$ a right angle. Can the parallels be constructed by any other method than that used in Ex. 1?

Ex. 3. Draw any $\angle BAC$. Bisect $\angle BAC$. From X any point in the bisector of $\angle BAC$ construct $XC \parallel AB$ and $XD \parallel AC$.

C Exercises

In the following exercises, you can prove the lines parallel by proving the alternate interior angles equal. Other methods may be possible in some cases. A variety of proofs is always interesting. A one-step analysis is sufficient.

1. Prove Th. 9 by using other pairs of corresponding angles than those shown in Fig. 117.

2. Prove Th. 9 by proving $\angle 3 = \angle 4$ (Fig. 121). (See Post. 16, p. 31.)

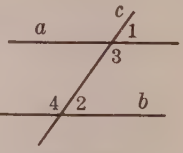


Fig. 121

3. Prove Th. 9 by the method given in Ex. 2, but use other pairs of corresponding angles.

4. If two straight lines in the same plane are cut by a third straight line so that the alternate exterior angles are equal, the two straight lines are parallel. Give two proofs, one based on Ax. 5, p. 30, the other on Post. 16, p. 31.

5. If, in Fig. 121, lines a and b are cut by line c so that $\angle 1 + \angle 4 = 2$ rt. $\measuredangle$, prove $a \parallel b$.

B Exercises

The analyses for the following exercises may require more than one step. Note the analysis for Ex. 1.

1. If two segments bisect each other, the segments joining their extremities are parallel.

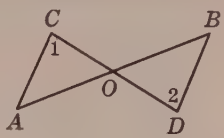


FIG. 122

Analysis (Fig. 122):

I. To prove $AC \parallel BD$, prove $\angle 1 = \angle 2$.

II. " " $\angle 1 = \angle 2$, " . . . . . .

In Fig. 123, prove $WX \parallel YZ$ if:

2. $\angle 1 = \angle 5$, RX bisects $\angle 1$, and SY bisects $\angle 5$.

3. $\angle 1 = 68°$, $\angle 4 = 30°$, $\angle 6 = 38°$.

4. $\angle 1 = \angle 5$ and $\angle 2 = \angle 7$.

5. $\angle 1 = \angle 5$ and $\angle 2 = \angle 8$.

6. $\angle 1 = \angle 5$ and $\angle 4 = \angle 8$.

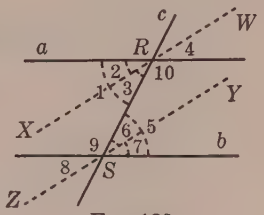


FIG. 123

7. If two straight lines are cut by a third straight line so that the exterior angles on the same side of the transversal are supplements, the lines are parallel.

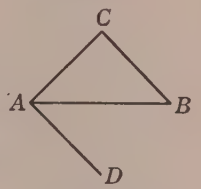


FIG. 124

8. Given (Fig. 124) $\triangle ABC$ isosceles and AB bisecting $\angle CAD$. Prove $AD \parallel BC$.

9. If the opposite sides of a four-sided figure are equal, they are also parallel.

10. Given (Fig. 125) $\angle A = \angle B$, $AC = BD$, and $AX = BY$. Prove $CY \parallel DX$.

11. Given (Fig. 125) $AX = BY$, $AC = BD$, and $CY = DX$. Prove $AC \parallel BD$ and $CY \parallel DX$.

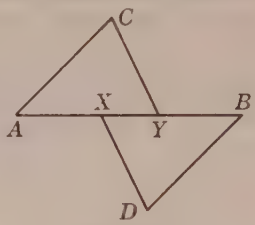


FIG. 125

113. Converse theorems. If two theorems are so related that the hypothesis and the conclusion of one are respectively the conclusion and the hypothesis of the other, each theorem is called the **converse** of the other.

For example: If two adjacent angles are supplementary, their outer sides are in a straight line.

Hypothesis: If two adjacent angles are supplementary,

Conclusion: Their outer sides are in a straight line.

For the converse we have: If the outer sides of two adjacent angles are in a straight line, the angles are supplementary.

It is important to note that the converse of a theorem is not always true. Often it may be proved true.

Ex. 1. State the converse of each of the following and tell whether or not each is true:

a. All opera singers are persons who have studied music.

b. A person born or naturalized in the United States is a citizen of the United States.

c. If the bisector of the vertex angle of a triangle is perpendicular to the base, the triangle is isosceles.

d. Vertical angles are equal.

e. Th. 4, page 58.

Ex. 2. State the converse of Th. 8, page 86; Th. 9, page 87; Th. 10, page 88; Th. 11, page 89. In the next group of theorems you will meet these converse theorems.

THEOREM 12*

114. If two parallel lines are cut by a third straight line, the alternate interior angles are equal.

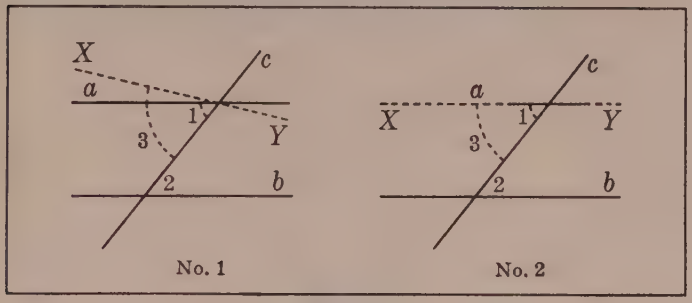


FIG. 126

Hypothesis: Line $a \parallel$ line b and line c cuts lines a and b.

Conclusion: $\angle 1 = \angle 2$.

Analysis and construction:

1. To prove $\angle 1 = \angle 2$, construct XY so that $\angle 3 = \angle 2$ and prove that $\angle 3 = \angle 1$.
2. To prove $\angle 3 = \angle 1$, prove XY coincides with line a.
3. To prove XY coincides with a, show that $XY \parallel b$ and $a \parallel b$.

Proof:

STATEMENTS	REASONS
1. (1) Line $a \parallel$ line b.	1. (1) Given.
(2) Line $XY \parallel$ line b.	(2) Why?
(3) $\therefore$ line XY coincides with line a.	(3) Post. 22, p. 84.
2. $\angle 3$ coincides with and equals $\angle 1$.	2. Their sides coincide.
3. $\angle 3 = \angle 2$.	3. Why?
4. $\therefore \angle 1 = \angle 2$.	4. $\angle 1$ coincides with an angle that is equal to $\angle 2$.

THEOREM 13

115. If two parallel lines are cut by a third straight line, the corresponding angles are equal.

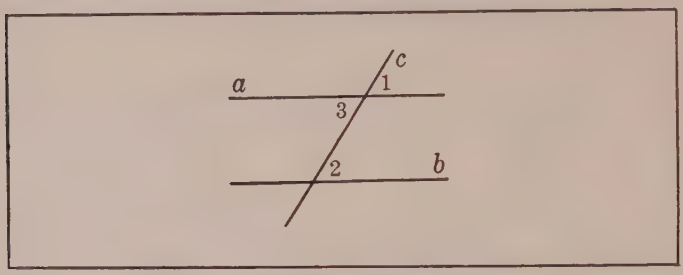


FIG. 127

Hypothesis: Line $a \parallel$ line b, and line c cuts lines a and b.

Conclusion: $\angle 1 = \angle 2$.

Analysis: To prove $\angle 1 = \angle 2$, show that $\angle 1 = \angle 3$ and $\angle 2 = \angle 3$.

Proof:

STATEMENTS	REASONS
1. $a \parallel b$.	1. Why?
2. $\therefore \angle 2 = \angle 3$.	2. Th. 12.
3. $\angle 1 = \angle 3$.	3. Why?
4. $\therefore \angle 1 = \angle 2$.	4. Why?

Ex. 1. Why are the proof and the analysis for Th. 12 so much harder than those for Th. 13?

Ex. 2. In Fig. 128 the lines are parallel as shown. If $\angle x = 115°$, find the number of degrees in each of the other angles.

Ex. 3. In Fig. 128, if the lines are parallel as shown, prove each of the following:

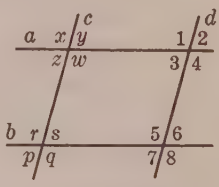


FIG. 128

a. $\angle x = \angle 4$.
b. $\angle y = \angle 6$.
c. $\angle w = \angle 5$.
d. $\angle 7 = \angle z$.
e. $\angle 3 = \angle s$.
f. $\angle 2 = \angle p$.

THEOREM 14

116. If two parallel lines are cut by a third straight line, the interior angles on the same side of the transversal are supplements of each other.

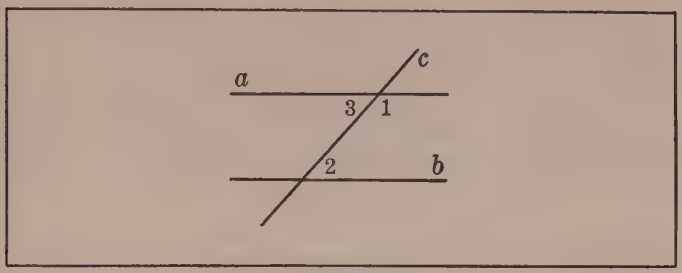


FIG. 129

Hypothesis: Line $a \parallel$ line b, and line c cuts lines a and b.

Conclusion: $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.

Analysis: To prove $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$, prove that $\angle 2 = \angle 3$ and that $\angle 3$ is the supplement of $\angle 1$.

Outline of proof. Give the reasons.

1. $a \parallel b$.
2. $\angle 2 = \angle 3$.
3. $\angle 3$ is the supplement of $\angle 1$.
4. $\therefore \angle 2$ is the supplement of $\angle 1$. ($\angle 2$ is equal to an angle that is the supplement of $\angle 1$.)
5. $\therefore \angle 2 + \angle 1 = 2$ rt. $\measuredangle$.

117. In the same plane non-parallel lines intersect. Two lines that are not in the same plane, however, may not intersect and still not be parallel. Hold two pencils to represent such lines. They are called **skew lines**.

There are, in general, three relative positions of lines: intersecting lines, parallel lines, and skew lines.

Exercise. Find in the room in which you are sitting illustrations of lines in each of the three relative positions.

THEOREM 15

118. A line which is perpendicular to one of two parallels is perpendicular to the other.

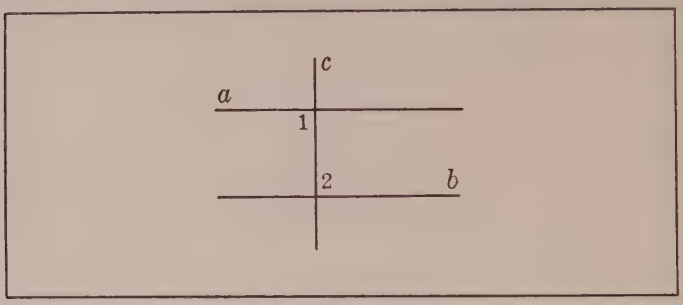


FIG. 130

Hypothesis: Line $a \parallel$ line b and line $c \perp$ line a.

Conclusion: Line $c \perp$ line b.

Analysis: To prove $c \perp b$, prove $\angle 2 = 1$ rt. $\angle$.

Outline of proof. Give the reasons.

1. $a \parallel b$.
2. $\therefore \angle 1 = \angle 2$.
3. $c \perp a$.
4. $\therefore \angle 1 =$ a rt. $\angle$.
5. $\therefore \angle 2 =$ a rt. $\angle$.
6. $\therefore c \perp b$.

119. Summary of facts concerning parallels:

A. If you wish to prove two lines parallel, prove one of the following: that —

(1) The alternate interior angles are equal.

(2) The corresponding angles are equal.

(3) The interior angles on the same side of the transversal are supplements.

(4) They are perpendicular to the same line.
(5) They are parallel to the same line.

B. If you know that two lines are parallel, you know that:

(1) The alternate interior angles are equal.
(2) The corresponding angles are equal.
(3) The interior angles on the same side of the transversal are supplements.
(4) A straight line perpendicular to one is perpendicular to the other.

C Exercises

1. By using pencils or rulers, illustrate the fact that, if you consider lines in space rather than on a plane, you can have more than one perpendicular to a line through a given point in the line.

2. If you consider lines in space, is Th. 11, p. 89, true? Illustrate by the use of pencils or rulers.

3. Answer question 2 for Th. 7, p. 85, and for Th. 15.

4. Prove Th. 13, p. 94, by using other pairs of corresponding angles than those shown in Fig. 127.

5. Prove Th. 13 by using $\angle 3$ and $\angle 4$ (Fig. 131). (See Post. 16, p. 31.)

6. Prove Th. 13 by the method indicated in Ex. 5 but using different pairs of corresponding angles.

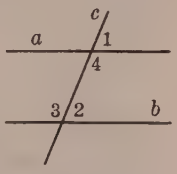


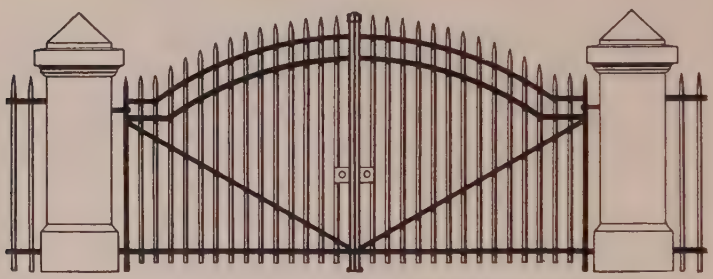
FIG. 131

7. If, in Fig. 131, $a \parallel b$, prove $\angle 1 + \angle 3 = 2$ rt. $\measuredangle$ (see analysis and proof for Th. 14, p. 95).

8. If two straight lines are cut by a third straight line so that a pair of alternate interior angles are equal, then a pair of corresponding angles are equal.

9. If two parallel lines are cut by a third straight line, the alternate exterior angles are equal.

Give two proofs, one using Ax. 5, p. 30, the other using Post. 16, p. 31.



From Stewart Iron Works Company

FIG. 132

10. Why are the crossbars used in Fig. 132? Note the parallel bars in the gates. What different kinds of angles are illustrated in the figure? Make a copy of part of this figure on the blackboard and indicate clearly where these angles are. Could the figure illustrate anything else?

11. Quote all the theorems that you have had that could be used to prove lines parallel. Which was used the most often?

B Exercises

1. *ABC* is any triangle. Through the vertices construct lines parallel to the opposite sides. Prove that each of the three triangles formed is congruent to the given triangle.

2. Any line parallel to the base of an isosceles triangle through the vertex bisects the exterior angle at the vertex.

3. If both pairs of opposite sides of a four-sided figure are parallel, they are also equal.

4. If a four-sided figure has one pair of sides equal and parallel, the other two sides are equal and parallel.

THEOREM 16

120. A. If two angles have their sides parallel right side† to right side and left side to left side, the angles are equal.

B. If two angles have their sides parallel right side to left side and left side to right side, the angles are supplementary.

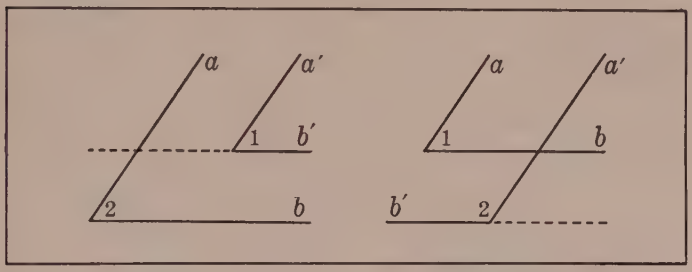


FIG. 133 FIG. 134

A. Hypothesis (Fig. 133): Line $a \parallel$ line a', line $b \parallel$ line b'.

Conclusion: $\angle 1 = \angle 2$.

Analysis: To prove $\angle 1 = \angle 2$, prove them each equal to a third angle.

Give the construction, name the third angle, and give the proof.

B. Hypothesis (Fig. 134): Line $a \parallel$ line a', line $b \parallel$ line b'.

Conclusion: $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.

Analysis: To prove $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$, prove $\angle 1$ equal to an angle that is the supplement of $\angle 2$.

Find the third angle and give the proof.

Exercise. Draw figures for Th. 16 with the angles in different positions from those shown in Figs. 133 and 134 and give proof.

† The right side of an angle is the right side as one stands in the angle and looks out.

C Exercises

Given (Fig. 135) $a \parallel b$, prove $c \parallel d$ if:

1. $\angle y = \angle 7$.
2. $\angle z = \angle 6$.
3. $\angle 4 = \angle r$.
4. $\angle w + \angle 7 = 2$ rt. $\measuredangle$.

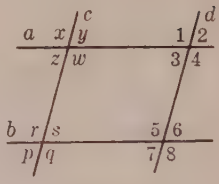


Fig. 135

Given (Fig. 135) $c \parallel d$, prove $a \parallel b$ if:

5. $\angle 1 = \angle r$.
6. $\angle 5 = \angle x$.
7. $\angle w + \angle 7 = 2$ rt. $\measuredangle$.

8. Given (Fig. 136) $AB \parallel CD$ and $AE \parallel CF$. Prove $\angle 1 = \angle 2$.

9. Given (Fig. 136) $AB \parallel CD$ and $\angle 1 = \angle 2$. Prove $AE \parallel CF$.

10. Given (Fig. 136) $\angle 1 = \angle 2$ and $AE \parallel CF$. Prove $AB \parallel CD$.

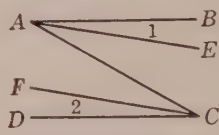


Fig. 136

11. A line $\parallel AB$ is to be drawn through X. A draftsman's triangle, ABC, is placed with one side on AB. The line AC is drawn and extended. The triangle is then moved to the position XYZ, with point A on point X (Fig. 137). XY is drawn. Why is $XY \parallel AB$?

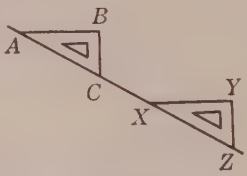


Fig. 137

12. How is a draftsman's T square used to draw parallel lines (Fig. 138)? Why are the lines parallel?

13. Show how to construct a line through a given point parallel to a given line by paper folding. Prove that your method is correct.

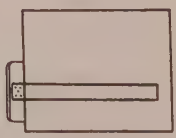


Fig. 138

B Exercises

1. Given $AC = BD$, $AC \parallel BD$, and $AX = BY$, prove $CY = DX$ (Fig. 139).

2. Using the data given in Ex. 1, prove $CY \parallel DX$.

3. Given $CY = DX$, $CY \parallel DX$, and $AX = BY$, prove $AC = BD$ (Fig. 139).

4. Using the data given in Ex. 3, prove $AC \parallel BD$.

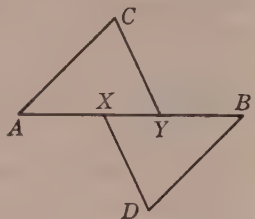


FIG. 139

5. If a segment between two parallel lines is bisected, any other segment between the parallels and through the point of bisection is also bisected by this point.

6. Given (Fig. 140) $h \parallel k$, $n \perp h$, and $m \perp k$, prove $n \parallel m$. (Show that Th. 15, p. 96, and Th. 11, p. 89, apply).

7. If (Fig. 140) $h \parallel k$, $n \perp h$, and $m \parallel n$, prove $m \perp k$. (Apply Th. 15, p. 96.)

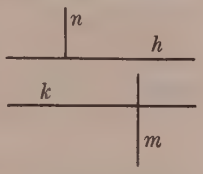


FIG. 140

8. If two parallel lines are cut by a third straight line, the exterior angles on the same side of the transversal are supplements.

9. If (Fig. 141) $AC = XZ$, $BC = YZ$, and $AX = YB$, prove $AC \parallel XZ$ and $BC \parallel YZ$.

10. If (Fig. 141) $AX = YB$, $AC \parallel XZ$, and $BC \parallel YZ$, prove $AC = XZ$ and $BC = YZ$.

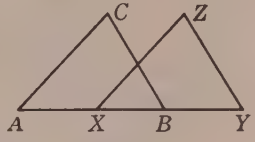


FIG. 141

11. Given (Fig. 142) $AW = BZ$, $AY = BX$, and $AW \parallel BZ$, prove $XZ \parallel YW$.

12. Given $AW \parallel BZ$, $AY = BX$, and $YW \parallel XZ$, prove $BZ = AW$ (Fig. 142).

13. Given $AY = BX$, $YW \parallel XZ$, and $YW = XZ$, prove $AW \parallel BZ$ and $AW = BZ$ (Fig. 142).

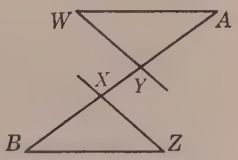


FIG. 142

14. Construct from a given point outside a line a ray that shall make a given angle with the given line. Is there more than one solution of the problem?

15. If, in Fig. 143, line $h \parallel$ line k and O is any point between the parallels, prove $\angle b = \angle a + \angle c$.

Hint. Draw a line through O parallel to h and k.

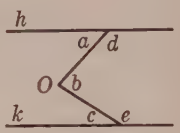


FIG. 143

16. If, in Fig. 143, line $h \parallel$ line k as in Ex. 15, prove $\angle d + \angle b + \angle e = 4$ rt. $\angle$s.

17. Prove Th. 7, p. 85, by drawing any oblique transversal and using Th. 8.

18. Prove Th. 7, p. 85, by constructing a line perpendicular to one of the given lines and proving it perpendicular to the others. (Angles are not to be used. Apply Th. 15, p. 96, and Th. 11, p. 89.)

19. If a four-sided figure has both pairs of opposite sides parallel and one angle a right angle, all of its angles are right angles.

20. If a four-sided figure has both pairs of opposite sides parallel, the opposite angles are equal.

21. If a four-sided figure has one pair of opposite sides equal and parallel, the segments joining opposite vertices bisect each other.

22. If two angles have their sides parallel right side to right side and left side to left side, the bisectors of the angles are parallel or in the same straight line.

23. In Fig. 144, $\angle BAC$ is any angle. O is any point on the ray AB. If the point X starts at point A and moves along the ray C indefinitely, what are the limiting values of the $\angle AOX$ and of the $\angle BOX$? Investigate the case in which the point X moves from A along AD.

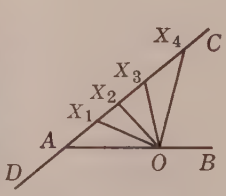


FIG. 144

TOPIC 3. SUMS OF ANGLES IN TRIANGLES AND POLYGONS

THEOREM 17*

121. The sum of the interior angles of a triangle is two right angles.

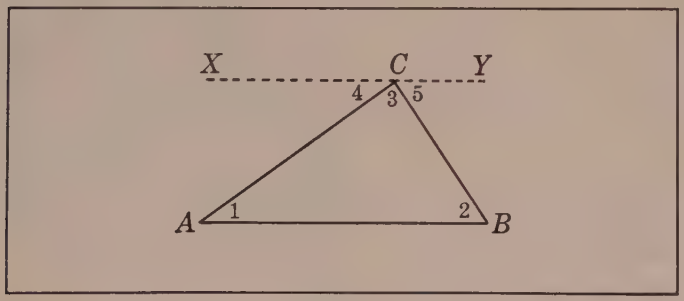


FIG. 145

Hypothesis: $\triangle ABC$ is any triangle.

Conclusion: $\angle 1 + \angle 2 + \angle 3 = 2$ rt. $\measuredangle$.

Analysis and construction:

1. To prove $\angle 1 + \angle 2 + \angle 3 = 2$ rt. $\measuredangle$, prove $\angle 1$, $\angle 2$, and $\angle 3$ equal to angles whose sum is two rt. $\measuredangle$.
2. Construct XY through $C \parallel AB$ and prove $\measuredangle 1$, 2, and 3 equal to $\measuredangle 4$, 5, and 3 respectively.

Outline of proof. Give the reasons.

1. $XY \parallel AB$.
2. $\angle 1 = \angle 4$.
3. $\angle 2 = \angle 5$.
4. $\angle 4 + \angle 3 + \angle 5 = 2$ rt. $\measuredangle$ (see Post. 19, p. 32).
5. $\therefore \angle 1 + \angle 2 + \angle 3 = 2$ rt. $\measuredangle$ (Ax. 6, p. 30).

Exercise. Verify Th. 17 by tearing off the corners at the vertices of the triangle and rearranging them.

122. Cor. I. **If two angles of one triangle are equal respectively to two angles of a second triangle, the third angles are equal.**

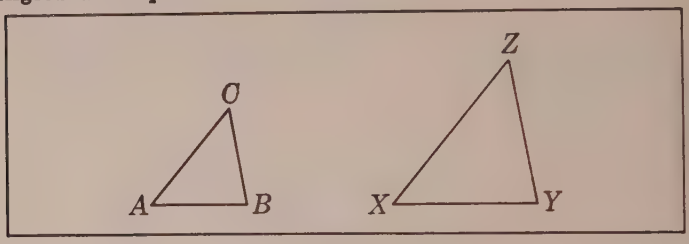


Fig. 146

Hypothesis: $\triangle ABC$ and $\triangle XYZ$ have $\angle A = \angle X$ and $\angle B = \angle Y$.

Conclusion: $\angle C = \angle Z$.

Analysis: To prove $\angle C = \angle Z$,

$$\text{prove} \begin{cases} \angle A + \angle B + \angle C = \angle X + \angle Y + \angle Z. \\ \angle A + \angle B = \angle X + \angle Y. \end{cases}$$

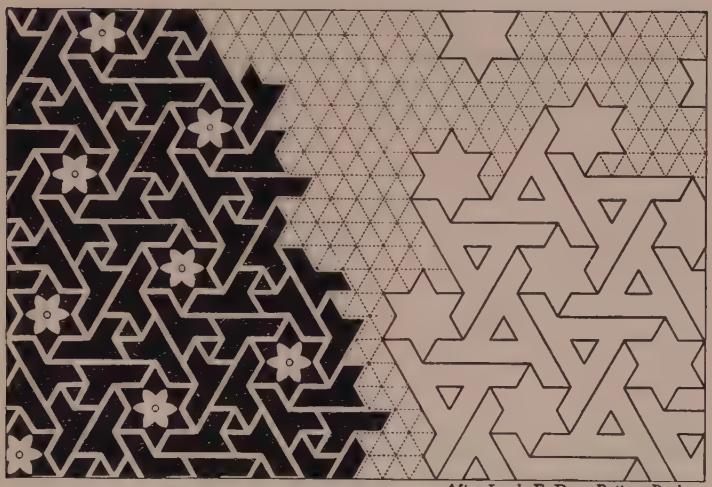
Outline of proof. Give the reasons.

1. *a.* $\angle A + \angle B + \angle C = 2$ rt. $\measuredangle$.
   b. $\angle X + \angle Y + \angle Z = 2$ rt. $\measuredangle$.
   c. $\therefore \angle A + \angle B + \angle C = \angle X + \angle Y + \angle Z$.
2. *a.* $\angle A = \angle X$.
   b. $\angle B = \angle Y$.
   c. $\therefore \angle A + \angle B = \angle X + \angle Y$.
3. *a.* $\angle A + \angle B + \angle C = \angle X + \angle Y + \angle Z$.
   b. $\angle A + \angle B = \angle X + \angle Y$.
   c. $\therefore \angle C = \angle Z$.

123. Cor. II. **Each angle of an equilateral triangle is 60°.**

Exercise. Construct with compasses, without the use of the protractor, angles of 30°, 15°, 75°, 7° 30′, 67° 30′, 165°, 150°.

124. Cor. III. **The acute angles of a right triangle are complements of each other.**



After Lewis F. Day, *Pattern Design*

FIG. 147. **Indian lattice pattern.** This design is built on a network of equilateral triangles.

C Exercises

1. Find the number of degrees in the third angle of a triangle if the other two angles are:

a. 46°, 52°. *c.* 57° 16′, 39° 31′. *e.* 19° 50′, 79° 21′.
b. 31°, 64° 15′. *d.* 36° 41′, 72° 10′. *f.* 75° 25′, 67° 40′.

2. Find the second acute angle of a right triangle if the first angle is 78°; 79° 51′; 46° 19′; 37° 37′; 60°; 45°.

3. If the vertex angle of an isosceles triangle is 68°, find the number of degrees in each base angle.

4. If one base angle of an isosceles triangle is 36°, find the number of degrees in the vertex angle.

5. Would it be possible to have a triangle whose angles are: (*a*) 90°, 30°, 60°? (*b*) 76°, 84°, 20°? (*c*) 135°, 32°, 22°? Why?

6. Give a set of angles (*a*) that could do for the angles of a triangle; (*b*) that could not do for the angles of a triangle.

Theorem 18

125. The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.

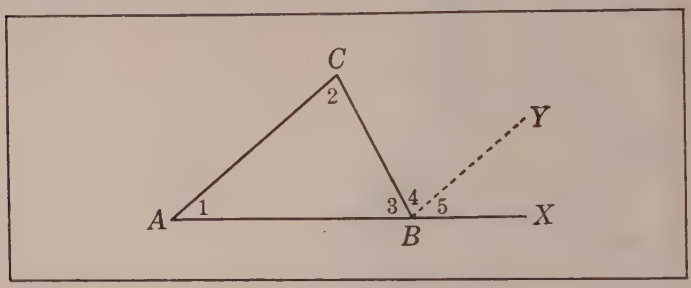


Fig. 148

Hypothesis: In $\triangle ABC$, the side AB is extended, forming the exterior $\angle XBC$.

Conclusion: $\angle XBC = \angle 1 + \angle 2$.

Analysis and construction:

1. To prove $\angle XBC = \angle 1 + \angle 2$, divide $\angle XBC$ into two parts and prove these parts equal to $\angle 1$ and $\angle 2$ respectively.
2. $\therefore$ draw BY through $B \parallel AC$ and prove $\angle 4 = \angle 2$ and $\angle 5 = \angle 1$.

Outline of proof. Give the reasons.

1. $BY \parallel AC$.
2. $\angle 5 = \angle 1$.
3. $\angle 4 = \angle 2$.
4. $\angle 4 + \angle 5$ or $\angle XBC = \angle 1 + \angle 2$ (Ax. 1, p. 29).

C Exercises

1. An exterior angle of a triangle is 145°; one of the opposite interior angles is 62°. Find the number of degrees in each of the angles of the triangle.

2. If the exterior angle at the vertex of an isosceles triangle is 128°, find the number of degrees in each of the angles of the triangle.

3. The base angles of an isosceles triangle are each 47°. Find the exterior angle at the vertex.

4. Why is a network of equilateral triangles possible (Fig. 147)?

5. Construct a triangle ABC with the angles unequal. Extend the sides in order, forming three unequal exterior angles. If the angles of the triangle are 73°, 63°, and 44°, find the number of degrees in each exterior angle. Can this be done in more than one way?

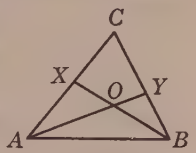


FIG. 149

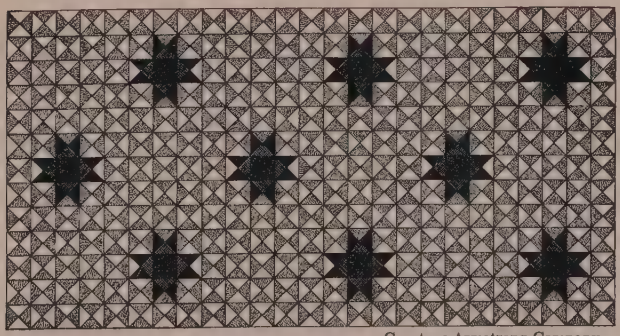
6. In Fig. 149 find as many illustrations as possible of an exterior angle of a triangle. Show how Th. 18 applies in each case.

7. Construct a triangle ABC with $\angle A = \angle B$. Extend BC to D, forming the exterior angle DCA. Prove $\angle DCA = 2\angle B$.

8. How many degrees in each angle of an isosceles right triangle? Note the isosceles right triangles in Fig. 150.

9. Construct an isosceles right triangle with one leg 8 cm.

NOTE. Problem 8, p. 171, may be studied here.



Courtesy Armstrong Company

FIG. 150. **Printed linoleum pattern.** This design is composed of isosceles right triangles.

B Exercises

1. Construct with the compasses a triangle whose base is 6 cm. and whose base angles are 45° and 60° respectively.

2. Draw an isosceles triangle ABC with AB as base. Draw AX from $A \perp BC$ and meeting BC at X. If $\angle C = 64°$, find the number of degrees in $\angle XAB$.

3. Draw an isosceles triangle ABC with AB as base. Draw AX bisecting $\angle A$ and meeting BC at X. If $\angle C = 110°$, find the number of degrees in $\angle AXB$.

4. Two base angles of $\triangle ABC$ are 74° and 40° respectively. If AX bisects $\angle A$ and BY bisects $\angle B$ and if AX and BY intersect at O, find the number of degrees in the angles at O.

5. Find the number of degrees in the angles made by the bisectors of two angles of an equilateral triangle.

6. If one angle of a triangle is double the second and the third is double the first, find the number of degrees in each of the angles of the triangle.

7. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

8. In Fig. 151, $\triangle ABC$ is isosceles. AO and OB bisect $\measuredangle A$ and B respectively. If the legs CA and CB are made to increase and to decrease in length, what is the upper limit to $\angle AOB$? What is the lower limit?

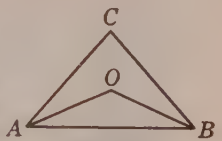


FIG. 151

9. In Fig. 152, $\triangle ABC$ is isosceles. AX and BY are perpendicular to BC and AC respectively. If the legs AC and BC are made to increase and to decrease in length, what is the upper limit to $\angle AOB$? What is its lower limit?

FIG. 152

A Exercises

NOTE. Th. 17, p. 103, is one of the most famous theorems of geometry. It is supposed that the ancient Greeks knew that it was true for equilateral and for isosceles triangles before they knew that it was true for all triangles. The proof given on page 103 is supposed to be that of Pythagoras (about 500 B.C.), and may be one of the earliest proofs for this theorem.

1. See if you can find in any books in your library any other proofs for Th. 17, p. 103.

2. Are there any other ways of putting in construction lines for Th. 17, p. 103, so that $\angle 1$, $\angle 2$, and $\angle 3$ may be proved equal to angles whose sum is two right angles? Give proof (see Fig. 153).

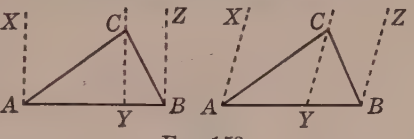


FIG. 153

3. How many degrees in each angle of an isosceles triangle if the vertex angle is (*a*) three times the sum of the base angles? (*b*) $\frac{1}{2}$ the sum of the base angles? (*c*) $\frac{1}{3}$ the sum of the base angles? (*d*) equal to the sum of the base angles? In each case construct the triangle from the results obtained.

4. If the vertex angle of an isosceles triangle is 62°, find the number of degrees that the bisector of one base angle makes with the opposite side.

5. If the vertex angle of an isosceles triangle is 42°, find the number of degrees in the angles at the intersection of the bisectors of the base angles. Find also the number of degrees in the angles at the intersection of the bisectors of the exterior angles at the base of the triangle.

6. If the vertex angle of an isosceles triangle is 70°, find the number of degrees in the angles at the intersection of the perpendiculars drawn to the equal sides from the extremities of the base.

THEOREM 19 (*Optional*)

126. A. If two angles have the sides of one perpendicular to the sides of the other, right side to right side and left side to left side, the angles are equal.

B. If two angles have the sides of one perpendicular to the sides of the other, right side to left side and left side to right side, the angles are supplementary.

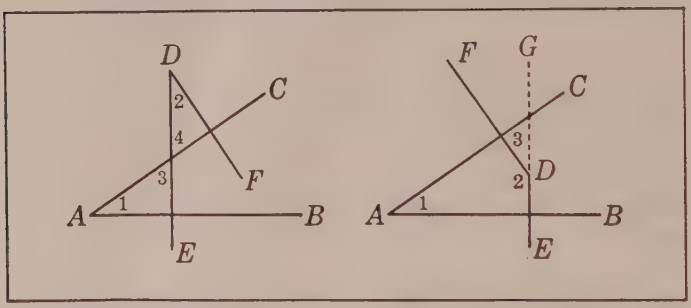


FIG. 154 FIG. 155

A. Hypothesis (Fig. 154): $\angle 1$ and $\angle 2$ have $AB \perp DE$ and $AC \perp DF$.

Conclusion: $\angle 1 = \angle 2$.

Analysis: To prove $\angle 1 = \angle 2$, show that $\angle 1$ and $\angle 2$ are complements of equal angles.

B. Hypothesis (Fig. 155): $\angle 1$ and $\angle 2$ have $AB \perp DE$ and $AC \perp DF$.

Conclusion: $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.

Analysis: To prove $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$, prove $\angle 1$ is equal to an angle that is the supplement of $\angle 2$.

Extend ED to intersect AC and apply Th. 19A.

Exercise. Prove Th. 19 using figures different from those given in Figs. 154 and 155.

127. Polygons† were defined on p. 44. A polygon has **sides, angles,** and **vertices.** Segments joining any two non-consecutive vertices are called **diagonals.** The sum of the sides is called the **perimeter** of the polygon.

128. Polygons are named from the number of sides:

a polygon of 3 sides is called a **triangle** or a **3-side**;
a polygon of 4 sides is called a **quadrilateral** or a **4-side**;
a polygon of 5 sides is called a **pentagon** or a **5-side**;
a polygon of 6 sides is called a **hexagon** or a **6-side**;
a polygon of 7 sides is called a **heptagon** or a **7-side**;
a polygon of 8 sides is called an **octagon** or an **8-side**;
a polygon of 10 sides is called a **decagon** or a **10-side**;
a polygon of 12 sides is called a **duodecagon** or a **12-side**;
a polygon of 15 sides is called a **pentadecagon** or a **15-side.**

129. A polygon with all of its sides and all of its angles equal is called a **regular polygon** (see Fig. 162, p. 114, also figures in Unit IX, p. 368).

130. Polygons may be **convex, concave,** or **cross.** See Fig. 156. Unless otherwise stated, use a convex polygon.

FIG. 156

Ex. 1. Find the sum of the four angles of a quadrilateral.

Ex. 2. Find the sum of the five angles of a pentagon (Fig. 157).

Suggestion. Divide the pentagon into triangles by drawing diagonals from one vertex and multiply the sum of the angles of one triangle by the number of triangles.

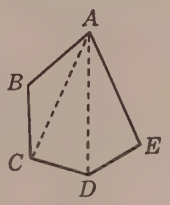


FIG. 157

Ex. 3. Find the sum of the angles of a hexagon; of an octagon; of a decagon.

† *To the teacher:* Study of angles in polygons, pp. 111–114, may be postponed.

THEOREM 20

131. The sum of the interior angles of a polygon of n sides is $2(n-2)$ right angles.

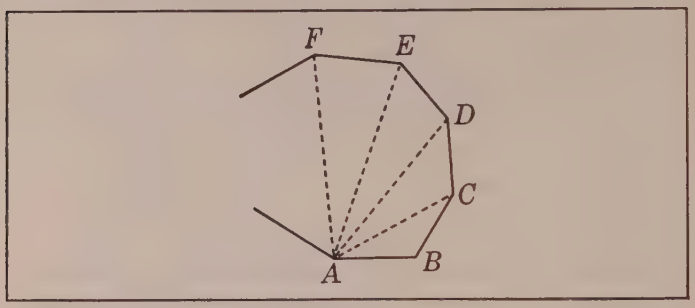


FIG. 158

Hypothesis: Polygon $ABCDEF$ etc. has n sides.

Conclusion: $\angle A+\angle B+\angle C+\text{etc.}=2(n-2)$ rt. $\measuredangle$.

Analysis and construction: To find the sum of the interior angles, $\angle A+\angle B+\angle C+$ etc., divide the polygon into triangles by drawing diagonals from one vertex and find the sum of all the angles of all the triangles.

Proof:

1. The diagonals divide the polygon into $(n-2)$ triangles.
2. The sum of the angles in each triangle is 2 rt. $\measuredangle$.
3. The sum of the angles in $(n-2)$ triangles is $2(n-2)$ rt. $\measuredangle$.

132. COR. **Each angle of a regular polygon of n sides is $\frac{2n-4}{n}$ rt. $\measuredangle$.**

Ex. 1. Find the sum of the interior angles of polygons 2 and 3 in Fig. 156.

Ex. 2. By substituting the proper number for n in the formula given in Th. 20, find the sum of the angles of a hexagon; of an octagon; of a 15-side; of a 16-side; of a 20-side; of a 24-side; of a 32-side.

Ex. 3. How many degrees in each interior angle of a regular octagon? of a regular pentagon? of a regular decagon? of a regular polygon of 12 sides? of 16 sides? of 20 sides? of 24 sides?

A Exercises

1. Prove Th. 20 by drawing the construction lines (*a*) from a point within the polygon; (*b*) from a point on one side of the polygon.

2. In Fig. 159, *ABCDE* is a convex polygon of five sides, all of whose angles are obtuse. The sides of the polygon are extended until they intersect, forming the star polygon shown. Find the sum of the angles in the points of the star. What would be the sum of the angles in the points of the star if the polygon had six sides? if it had eight sides?

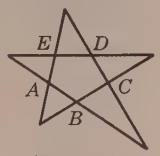


FIG. 159

3. Suppose the convex polygons used in Ex. 2 were regular, how many degrees would there be in the angle at any point of the star polygons formed?

FIG. 160

NOTE. The star made from the regular hexagon is very common (Fig. 160). It seems to have been an ancient symbol of Deity. It is on all the Jewish synagogues, is in many common trademarks, and is sometimes used as the policeman's star.

The star made from the regular pentagon (Fig. 161) is the star in the United States Flag. It is sometimes called the pentagram star. The pentagram star was the secret sign of a brotherhood or secret society of mathematicians who had their home in southern

FIG. 161

Italy. They called this star *health*. They were called Pythagoreans because their society was founded by Pythagoras, a Greek who settled there. Pythagoras died about 501 B.C. See if you can draw an irregular pentagram star without taking your pencil from the paper. You will find some good stories about the Pythagoreans in the books on the history of mathematics in the libraries.

4. Why can equilateral triangles, squares, or regular hexagons be used by themselves for tiled floors while regular pentagons cannot be so used?

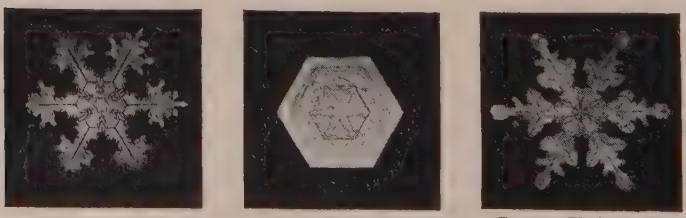
NOTE. The use of the polygons in Ex. 4 above for tiles probably dates back to the ancient Egyptians. Bring in any floor patterns that you can find that are made entirely of regular polygons.

5. Show that tiled floors can be made of octagons and squares.

6. *The sum of the exterior angles of a polygon of n sides is 4 rt. ∠.*

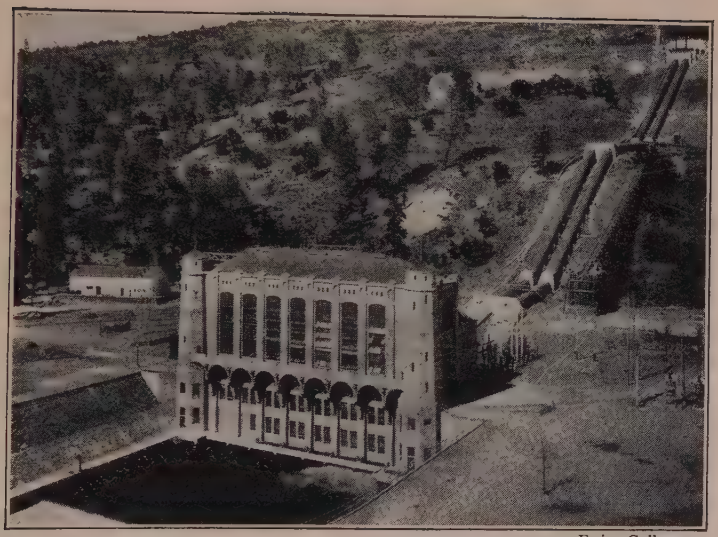
Suggestion. Show that the sum of the exterior and interior angles is $2n$ rt. ∠. Subtract from this the sum of the interior angles as found in Th. 20.

7. It is possible to have a regular polygon each of whose exterior angles is 24°; 36°; 45°. How many sides would each polygon have?



Courtesy W. A. Bentley

FIG. 162. **Some typical forms of snow crystals.** These forms are derived from regular hexagons.



Ewing Galloway

FIG. 163. **The power plant of a great hydroelectric plant in Shasta County, California.** In the parallel arrangement of the pipes carrying water down the mountain side, as well as in the architecture of the building, geometric principles have been used.

Survey Test

1. (*a*) What postulates are used to locate a line definitely? (*b*) In what other way than by two points is a line located?

2. (*a*) What is a transversal? (*b*) Name the kinds of angles formed when a transversal intersects two parallel straight lines. Illustrate with a figure.

3. (*a*) How is an exterior angle of triangle formed? (*b*) What are non-adjacent interior angles of a triangle? (*c*) Is the exterior angle of a triangle greater or less than or equal to either of the non-adjacent interior angles?

4. What can you say of a whole in relation to any of its parts?

5. If the first quantity is greater than the second and the second is greater than the third, is the first greater or less than the third?

6. (*a*) What do you understand by an indirect proof? (*b*) By what other name is an indirect proof sometimes called? (*c*) What is the general procedure in using an indirect proof?

7. (*a*) Define parallel lines. (*b*) Quote the fundamental postulate concerning parallels. (*c*) Upon what do the proofs for Theorems 7 and 8 concerning parallels depend?

8. (*a*) Quote the theorems that may be used to prove lines parallel. (*b*) Which of these statements would be true if the lines were not all in the same plane? (*c*) Illustrate by using three rulers or pencils.

9. (*a*) In the same plane what is true of two straight lines that are perpendicular to the same straight line? (*b*) State the corollary of this theorem. Does this recall to your mind a postulate that you learned earlier in this book? If so, find it in the book.

10. Quote two theorems by means of which you can draw a straight line through a given point parallel to a given straight line.

11. (*a*) What is the converse of a theorem? (*b*) Why must the converse of a theorem be proved before being accepted?

12. If two parallel lines are cut by a third straight line, (*a*) what angles are equal? (*b*) What angles are supplementary? Illustrate in each case by a figure.

13. (*a*) What do you understand by skew lines? (*b*) What are, in general, the relative positions of two lines in the same plane? of two lines not in the same plane? Illustrate with pencils or rulers.

14. What theorem concerning parallels may be used to prove lines perpendicular?

15. (*a*) Quote the theorems concerning parallels which were proved by indirect proofs. (*b*) Why was this necessary?

16. (*a*) Quote the theorem concerning angles with their sides parallel. (*b*) What is meant by the right side of an angle?

17. What is the sum of the interior angles of a triangle?

18. (*a*) If two angles of one triangle are equal respectively to two angles of a second triangle, are the triangles congruent? Why? (*b*) What can you say of the triangles?

19. (*a*) How many degrees in each angle of an equilateral triangle? (*b*) Why?

20. (*a*) What is the sum of the acute angles of a right triangle? (*b*) Why?

21. The sum of what two angles in a triangle is equal to the exterior angle of the triangle? Illustrate with a figure.

22. (*a*) Define a polygon. (*b*) What is the diagonal of a polygon? the perimeter?

23. By what names are polygons of three, four, five, six, seven, eight, ten, twelve, and fifteen sides called?

24. (*a*) Define a regular polygon. (*b*) Draw a convex polygon; a concave polygon; a cross polygon.

25. What is the sum of the interior angles of a quadrilateral? Why?

26. State the formula (*a*) for the sum of the interior angles of a polygon; (*b*) for each angle of a regular polygon.

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	81	10	90	19	104
2	82	11	92	20	104
3	82, 83	12	93–95	21	106
4	82	13	95	22	111
5	82	14	96	23	111
6	84, 85, 89	15	85, 86, 93	24	111
7	84–86	16	99	25	111
8	96, 97	17	103	26	112
9	89, 31	18	104		

TOPIC 4. CONGRUENT TRIANGLES II

133. Practical uses of congruent triangles. You have already learned something of the uses made of congruent triangles, especially in measuring distances across streams or ponds (pages 55 and 56) and in construction work (page 69). For some of the uses of the theorems in this topic see pages 126 and 127.

134. If pages 73–78 were omitted when Topic 1 was studied, they should be studied at this time.

Ex. 1. Quote the theorems used in proving triangles congruent.

Ex. 2. For what purposes are congruent triangles used?

Ex. 3. If you have an analysis that says:

To prove $AB = XY$, prove $\triangle ABO \cong \triangle XYZ$

what would be the reason that would prove $AB = XY$?

Ex. 4. Find two exercises that you had in congruent triangles in Topic 1, pp. 41–80, in which it is required to prove one line perpendicular to another. Draw the figure and give the hypothesis, conclusion, and analysis for each.

Ex. 5. Find two exercises in which it is required to prove that an angle is bisected and follow the instructions given in Ex. 4.

Ex. 6. Find two exercises in which it is required to prove a triangle isosceles and follow the instructions given in Ex. 4.

Ex. 7. Find two exercises in B Exercises, p. 76, that have an analysis of more than two steps. Work these exercises, giving the analysis and proof in full.

Ex. 8. Find an exercise in B Exercises, p. 67, that makes use of a pair of overlapping triangles. Work this exercise, giving the analysis and proof in full.

Ex. 9. Find in Topic 2, pp. 81–102, two exercises in which it is required to prove two lines parallel and for which a two-step analysis is necessary. Follow the directions in Ex. 4.

THEOREM 21*

135. If two angles of a triangle are equal, the triangle is isosceles.

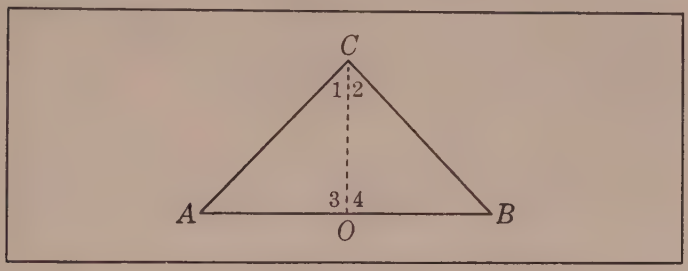


FIG. 164

Hypothesis: $\triangle ABC$ has $\angle A = \angle B$.

Conclusion: $\triangle ABC$ is isosceles.

Analysis:

1. To prove $\triangle ABC$ isosceles, prove $AC = BC$.
2. " " $AC = BC$, { draw CO bisecting $\angle C$, prove $\triangle AOC \cong \triangle BOC$.

Outline of proof. Give the reasons.

1. $\angle A = \angle B$.
2. $\angle 1 = \angle 2$.
3. $\therefore \angle 3 = \angle 4$ (§122, p. 104).
4. $CO = CO$.
5. $\therefore \triangle AOC \cong \triangle BOC$.
6. $\therefore AC = BC$.
7. $\therefore \triangle ABC$ is isosceles.

C Exercises

Use Th. 21 in the exercises that follow.

1. If two angles of a triangle are 80° and 50° respectively, the triangle is isosceles.

2. If one angle of an isosceles triangle is 60°, the triangle is equilateral.

3. Given (Fig. 165) AB bisecting $\angle CAD$ and $BC \parallel AD$, prove $\triangle ABC$ isosceles.

4. The lines bisecting the base angles of an isosceles triangle form with the base a second isosceles triangle.

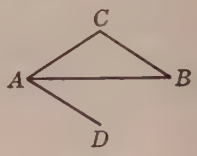


FIG. 165

5. The bisectors of the exterior angles at the base of an isosceles triangle form with the base a second isosceles triangle.

6. A line cutting the base and one leg of an isosceles triangle and parallel to the other leg cuts off a second isosceles triangle.

7. If, in Fig. 166, $\triangle ABC$ is isosceles and $DE \parallel BC$, prove $\triangle DEA$ isosceles.

8. A line parallel to the base of an isosceles triangle and cutting both legs of a triangle cuts off a second isosceles triangle.

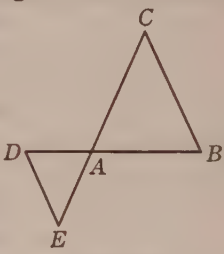


FIG. 166

9. Prove Ex. 8 if the line cuts the legs of the triangle extended (*a*) beyond the base, (*b*) through the vertex.

136. This topic includes two special theorems on **congruent right triangles.**

Construct a right triangle with

Ex. 1. The hypotenuse equal to 5 cm. and one angle equal to $22\frac{1}{2}°$.

Ex. 2. The hypotenuse equal to a given segment and one acute angle equal to a given angle.

Ex. 3. The hypotenuse equal to 7 cm. and one leg equal to 4 cm.

THEOREM 22

137. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.

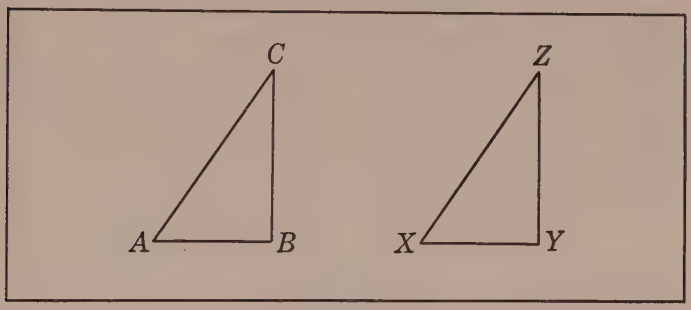


FIG. 167

Hypothesis: $\triangle ABC$ and $\triangle XYZ$ have $AC = XZ$, $\angle A = \angle X$, and $\angle B$ and $\angle Y$ right angles.

Conclusion: $\triangle ABC \cong \triangle XYZ$.

Analysis: To prove $\triangle ABC \cong \triangle XYZ$, prove $\angle C = \angle Z$.

Outline of proof. Give the reasons.

1. $\angle B = \angle Y$.
2. $\angle A = \angle X$.
3. $\therefore \angle C = \angle Z$.
4. $AC = XZ$.
5. $\triangle ABC \cong \triangle XYZ$.

Ex. 1. Perpendiculars from the mid-points of the equal sides of an isosceles triangle to the base are equal.

Ex. 2. Perpendiculars from the mid-point of the base of an isosceles triangle to the legs are equal.

Ex. 3. Perpendiculars from any point in the bisector of an angle to the sides of the angle are equal.

THEOREM 23*

138. Two right triangles are congruent if the hypotenuse and a side of one are equal to the hypotenuse and a side of the other.

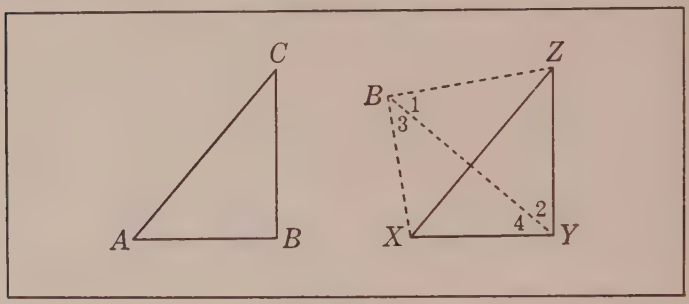


FIG. 168

Hypothesis: $\triangle ABC$ and $\triangle XYZ$ have $AC = XZ$, and $BC = YZ$, and $\angle B$ and $\angle Y$ rt. $\measuredangle$.

Conclusion: $\triangle ABC \cong \triangle XYZ$.

Analysis and construction:

1. To prove $\triangle ABC \cong \triangle XYZ$, prove $AB = XY$.
2. " " $AB = XY$, place $\triangle ABC$ so that AC, the hypotenuse of $\triangle ABC$, falls on XZ, the hypotenuse of $\triangle XYZ$, A on X, C on Z, and B opposite Y. Draw BY and prove $\angle 3 = \angle 4$.

Outline of proof. Give the reasons.

1. *a.* $BZ = YZ$.
   b. $\therefore \angle 1 = \angle 2$.
   c. $\angle B = \angle Y$.
   d. $\therefore \angle 3 = \angle 4$.
2. $\therefore XB = XY$.
3. $\therefore AB = XY$.
4. $\therefore \triangle ABC \cong \triangle XYZ$ (Th. 4, p. 58).

139. Cor. **The perpendicular from the vertex to the base of an isosceles triangle bisects the base.**

140. The perpendicular from any vertex of a triangle to the opposite side is called an **altitude** of the triangle. In an isosceles triangle two sides are equal and the third side is called the **base** of the triangle. The altitude on the base of an isosceles triangle is called the **altitude of the isosceles triangle.** The corollary above may then read, **The altitude of an isosceles triangle bisects the base.**

141. Two methods are given that may be used to prove that a **given triangle is isosceles**; prove either

(1) That two sides are equal, or
(2) That two angles are equal.

Five methods are given that may be used to prove **triangles congruent.**

Any two triangles are congruent if
(1) Two sides and the included angle, etc.
(2) Two angles and the included side, etc.
(3) Three sides, etc.

Two right triangles are congruent if
(1) The hypotenuse and an acute angle, etc.
(2) The hypotenuse and a side, etc.

B Exercises

Note. In the exercises below try the various methods suggested in §141. Use whichever method seems shortest and easiest.

1. In Fig. 169, $\triangle ABC$ is isosceles with $CA = CB$. If $CE = CD$ and AD and BE intersect at O, prove $\triangle AOB$ isosceles.

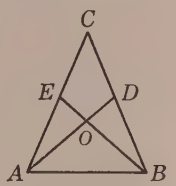


Fig. 169

2. Perpendiculars from the ends of the base of an isosceles triangle to the opposite sides are equal.

3. If the ray which is drawn through the vertex of a triangle parallel to the base bisects the exterior angle at the vertex, the triangle is isosceles.

4. In Fig. 170, $CO \perp h$. CA and CB are drawn from point C, so that $\angle 1 = \angle 2$ and they intersect line h at points A and B. A and B are joined with D any point in OC extended. Prove $\triangle ABD$ isosceles.

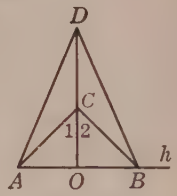


FIG. 170

5. Prove Ex. 4 if D is in CO extended.

6. In any $\triangle ABC$, perpendiculars from points A and B to the median to the side AB are equal. For what kind of triangle would your proof be meaningless?

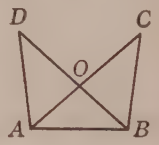


FIG. 171

7. If, in Fig. 171, $AC = BD$ and $AD = BC$, prove $\triangle AOB$ isosceles.

8. A line from the center of a circle perpendicular to a chord bisects the chord.

NOTE. A **chord** of a circle is a segment joining two points on the circle.

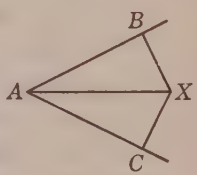


FIG. 172

9. In Fig. 172, $XB \perp AB$, $XC \perp AC$, and $XB = XC$. Prove AX bisects $\angle A$.

10. If, in an isosceles right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, it divides the triangle into two isosceles right triangles.

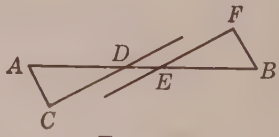


FIG. 173

11. Given (Fig. 173), $AD = BE$, $DC \parallel EF$, $AC \perp DC$, and $BF \perp EF$. Prove $AC = BF$.

12. If perpendiculars from the mid-point of the base of a triangle to the opposite sides are equal, the triangle is isosceles.

13. If perpendiculars from the ends of the base of a triangle to the opposite sides are equal, the triangle is isosceles.

14. In Fig. 174, $\triangle ABC$ is isosceles, $AX = BY$, and $CW = CZ$. Extend XW and YZ to meet at O. Prove $\triangle XOY$ isosceles.

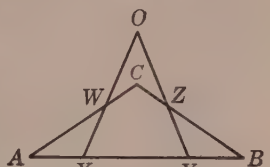


FIG. 174

15. If, in Fig. 174, $\triangle XOY$ is isosceles, $XA = YB$, and $XW = YZ$, prove $\triangle ABC$ isosceles.

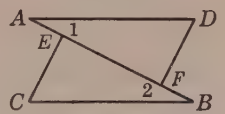


FIG. 175

16. Given (Fig. 175) AB any segment, $\angle 1 = \angle 2$, and $AD = BC$. From D and C perpendiculars are drawn to AB. Prove $DF = CE$.

A Exercises

1. In Fig. 176, $BD \parallel XY$, AB bisects $\angle XAC$, and AD bisects $\angle YAC$. Prove that $BC = CD$.

2. If from any point in the base of an isosceles triangle lines are drawn parallel to the legs, a four-sided figure is formed whose perimeter is equal to the sum of the legs of the triangle.

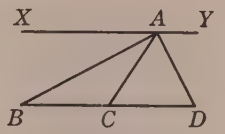


FIG. 176

3. In Fig. 177, AO bisects $\angle A$, BO bisects $\angle B$, and XY is drawn through $O \parallel AB$. Prove that $XY = AX + BY$.

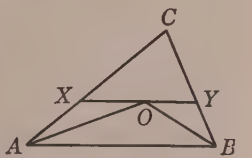


FIG. 177

4. Two triangles are congruent if two sides and the altitude on one of these sides are equal respectively to two sides and the corresponding altitude of the other.

5. If the bisectors of the base angles of an isosceles triangle meet the opposite sides at points D and E, prove that DE is parallel to the base.

6. In Fig. 178, ABC is an isosceles triangle with D any point in AC extended. From D a perpendicular is drawn to AB cutting CB at E. Prove that CDE is an isosceles triangle.

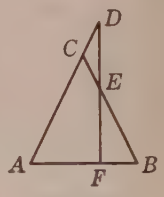


FIG. 178

7. Prove Ex. 6 when point F is on AB extended.

8. Show that a sailor may calculate his distance from a lighthouse as follows: In Fig. 179, C represents the lighthouse and BA the direction in which the boat is moving. With the proper instruments he may take the angle CBA. Then, if he notices the moment when the angle ($\angle CAD$) which the lighthouse makes with his course is twice $\angle B$, he can find the length of CA, for he can obtain the length of BA from his log.

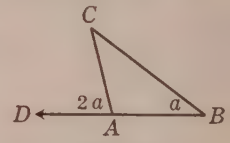


FIG. 179

9. In surveying, it is often necessary to run a line, as CD (Fig. 180), so that it will be in the same straight line with LA but on the other side of some obstacle to vision, such as a house or wood. Show that the following method will give the desired result: Lay off the $\angle a$ so as to clear the obstacle. Take BA, a convenient distance. Lay off $\angle b = 2\angle a$. Make $BC = BA$. Lay off $\angle c = \angle a$.

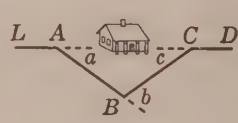


FIG. 180

10. If, in Fig. 181, OC is the perpendicular bisector of AB, what are the limiting values of the length of a perpendicular from O to the segment BX as X moves along the ray OC?

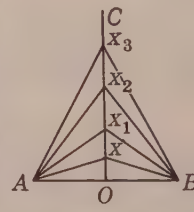


FIG. 181

11. Fig. 182 shows a regular hexagon $ABCDEF$. If alternate sides are extended in both directions as indicated, prove that a regular triangle XYZ is formed.

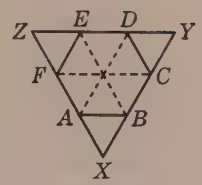


FIG. 182

NOTE. Tiled and mosaic floors are often made of equilateral triangles so colored that various patterns are formed. Fig. 183 shows one such design. See Ex. 4 and note, p. 114.

FIG. 183

12. In Fig. 184, ABC is an isosceles right triangle with $\angle C$ a right angle. CD is perpendicular from C to AB. Construct $XY \parallel AC$ and $= YC$ and $YZ = YC$.

NOTE. The patterns used in applied design are usually made by repeating at regular intervals some very simple figure called the unit. Figs. 185 and 186 show two parquet floor patterns. Show how Fig. 184 in whole or in part is used in each of these designs.

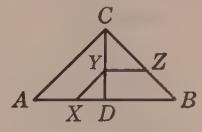


FIG. 184

FIG. 185

FIG. 186

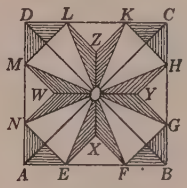


FIG. 187

Parquet floor designs

13. In Fig. 187, $ABCD$ is a four-sided figure with its sides equal and its angles right angles. $AE = BF = BG = CH =$ etc. EX and HY are parallel to AC; XF and MW are parallel to DB. NW, KZ, GY, and LZ are similarly drawn. Prove that $\triangle NOE$, EOF, and EXF are isosceles. How many isosceles triangles does the figure contain? What triangles in the figure are congruent? Draw HE and prove $EH \parallel AC$.

Survey Test

1. If two angles of a triangle are equal, what kind of triangle is the triangle?

2. In proving exercises by congruent triangles, when may you use a one-step analysis? a two-step analysis?

3. (*a*) Quote two theorems in the proof of which §122, page 104, is used. Show where it is used in each case. (*b*) How is Th. 21 used in the proof of Th. 23?

4. Define an altitude of a triangle.

5. What is considered the altitude of an isosceles triangle?

6. How does the altitude of an isosceles triangle affect the base?

7. By what two methods can you prove that a triangle is isosceles?

8. Quote five theorems which may be used in proving two triangles congruent.

9. Quote the theorems by which you can prove two right triangles congruent.

10. Name three important properties of an isosceles triangle.

11. Name three places in the world about you where you will find triangles.

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*
1	119	7	123
2	53, 66	8	123
3	119, 121, 122	9	123
4	123	10	151
5	123	11	41, 42
6	123		

TOPIC 5. SPECIAL QUADRILATERALS

142. Some of the ways in which quadrilaterals are used. A four-sided figure is called a **quadrilateral.** You will have no difficulty in citing numerous occurrences of such figures. All the kinds of quadrilaterals defined on these pages are extensively used in industry and ornament, as in the floor covering designs on pages 131, 139, and 141, the parallel rulers on page 139, and the adjustable bracket on page 140.

Unless otherwise stated, a quadrilateral should be drawn with no sides equal and no sides parallel.

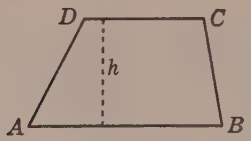


FIG. 188

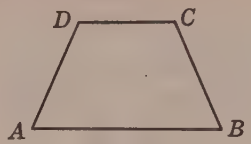


FIG. 189

143. A quadrilateral with but one pair of parallel sides is called a **trapezoid** (Fig. 188). If a trapezoid has its non-parallel sides equal, it is called an **isosceles trapezoid** (Fig. 189). The non-parallel sides of a trapezoid are sometimes called the **legs** of the trapezoid. The parallel sides of a trapezoid are called the **bases.**

144. A quadrilateral with two pairs of parallel sides is called a **parallelogram.** In Fig. 190, AB and CD, also AD and BC are called **opposite sides.** AB and AD are called **consecutive sides**; $\angle DAB$ and $\angle DCB$ are called **opposite angles**; $\angle DAB$ and $\angle ADC$ are called **consecutive angles**; AC and BD are called **diagonals.** While any side may be considered the base, AB and DC are usually called the **bases.**

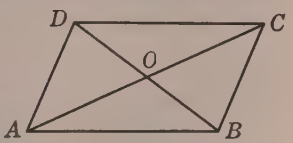


FIG. 190

145. The perpendicular distance between the bases of a trapezoid or of a parallelogram is called its **altitude.** (In Fig. 191, No. 1, h and h' are the altitudes.)

146. Special kinds of parallelograms. A parallelogram with one right angle is called a **rectangle** (Fig. 192, No. 2).

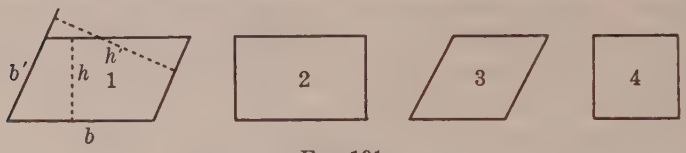


FIG. 191

A parallelogram with two consecutive sides equal is called a **rhombus** (Fig. 191, No. 3).

A parallelogram with two consecutive sides equal and one angle a right angle is called a **square** (Fig. 191, No. 4).

147. Construction of parallelograms. There is more than one way to construct a parallelogram. Until Ths. 26 and 27, p. 136, are studied, however, the parallelogram should be constructed with its opposite sides parallel.

Ex. 1. Show how to construct a parallelogram given two sides and the included angle (Fig. 192). (Use compasses.)

Suggestion. From D draw a line parallel to AB. From B draw a line parallel to AD. Let these lines intersect at C. $ABCD$ is the parallelogram. Why?

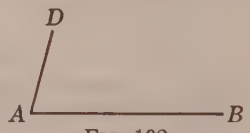
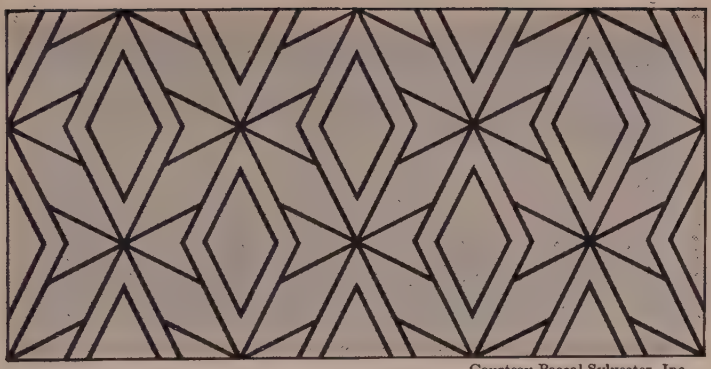


FIG. 192

Ex. 2. Construct with ruler and compasses (a) a rectangle; (b) a rhombus; (c) a square.

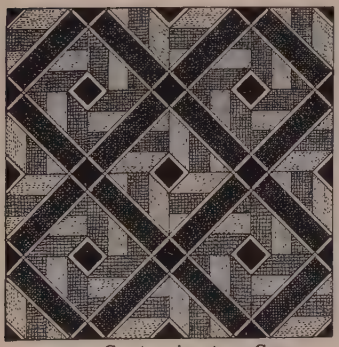
Ex. 3. Find in Fig. 193 illustrations of the different kinds of quadrilaterals mentioned so far in this chapter.

Ex. 4. Name three other places in the world about you where any of these figures are found.



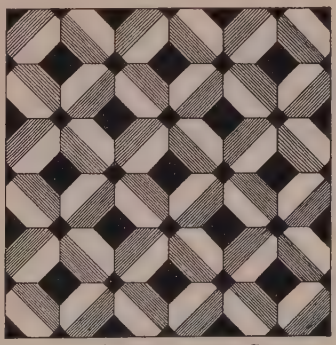
Courtesy Pascal Sylvester, Inc.

No. 1. Orna-metal brass strips for terrazzo floors



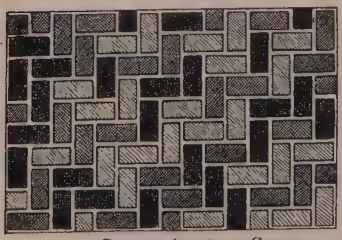
Courtesy Armstrong Company

No. 2. Printed linoleum



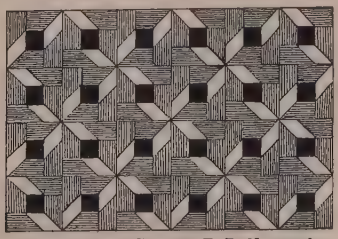
Courtesy Armstrong Company

No. 3. Molded inlaid linoleum



Courtesy Armstrong Company

No. 4. Handcraft tile inlaid linoleum



Courtesy E. R. Newcomb

No. 5. Parquet floor design

FIG. 193. **Floor coverings showing different kinds of quadrilaterals**

THEOREM 24

148. Each diagonal of a parallelogram divides it into two congruent triangles.

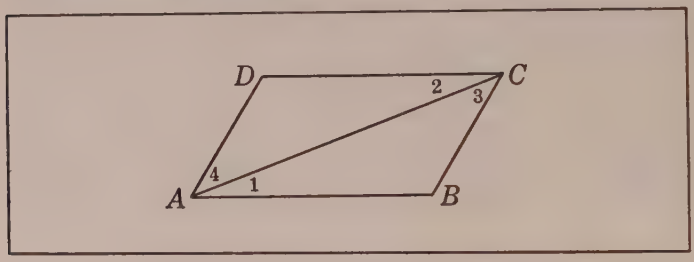


FIG. 194

Hypothesis: $ABCD$ is a $\square$ with the diagonal AC.

Conclusion: $\triangle ABC \cong \triangle ADC$.

Outline of proof. Give the reasons.

1. $AB \parallel DC$.
2. $\therefore \angle 1 = \angle 2$.
3. $AD \parallel BC$.
4. $\therefore \angle 3 = \angle 4$.
5. $AC = AC$.
6. $\therefore \triangle ABC \cong \triangle ADC$.

149. COR. I. **The opposite sides of a parallelogram are equal.**

150. COR. II. **Segments of parallel lines intercepted between parallel lines are equal.**

Given (Fig. 194) $AB \parallel DC$ and $AD \parallel BC$.

To prove $AD = BC$, prove $ABCD$ a $\square$ and apply Cor. I.

151. COR. III. **Segments of perpendiculars intercepted between parallel lines are equal.**

Draw the figure. Use the analysis to Cor. II.

152. Cor. IV. **All the sides of a rhombus are equal.**

See the definition of a rhombus and apply Cor. I.

153. Cor. V. **The opposite angles of a parallelogram are equal.**

Several proofs are possible.

1. Apply Th. 24 (p. 132).
2. By the use of Th. 8, p. 86, and Ax. 1, p. 29.
3. By a method similar to that used for Th. 16, p. 99.

Ex. 1. Two consecutive angles of a parallelogram are supplementary.

Ex. 2. The sum of the angles of a parallelogram is four right angles.

154. Cor. VI. **All the angles of a rectangle are right angles.**

See definition of a rectangle and apply Th. 15, p. 96, and §153, above.

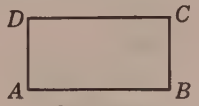


Fig. 195

155. The diagonals of a parallelogram differ from the diagonals of other four-sided figures in important respects.

Exercise. Fig. 196 shows a parallelogram, a trapezoid, and two other quadrilaterals. Which ones might be called convex quadrilaterals? Which one may be said to have a reëntrant angle? In which cases do the diagonals intersect? In which case do they appear to bisect each other?

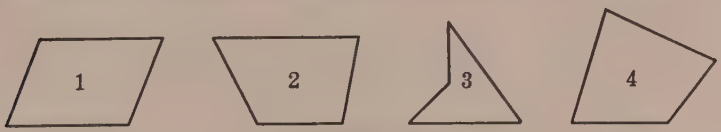


Fig. 196

It is assumed that the diagonals of a convex quadrilateral intersect. It will now be proved that the diagonals of a parallelogram bisect each other.

Theorem 25

156. The diagonals of a parallelogram bisect each other.

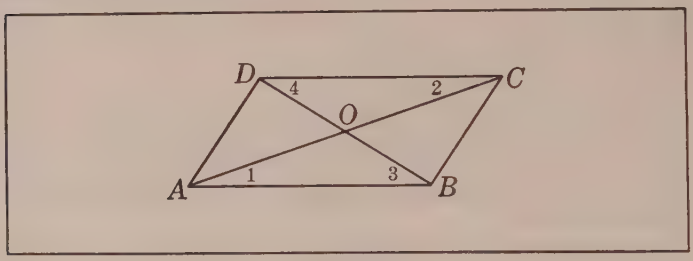


Fig. 197

Hypothesis: $ABCD$ is a $\square$ with the diagonals AC and BD intersecting at O.

Conclusion: AC and BD bisect each other.

Analysis:

1. To prove that AC and BD bisect each other, prove $AO = OC$ and $DO = OB$.
2. To prove $\left\{\begin{matrix} AO = OC \\ DO = OB \end{matrix}\right\}$, prove $\triangle AOB \cong \triangle DOC$.

Outline of proof. Give the reasons.

1. $AB \parallel DC$.
2. $\angle 1 = \angle 2$.
3. $\angle 3 = \angle 4$.
4. $AB = DC$.
5. $\therefore \triangle AOB \cong \triangle DOC$.
6. $\therefore AO = OC$ and $DO = OB$.

C Exercises

1. Construct a parallelogram with
   a. $\angle A = 60°$ and the including sides 5.3 cm. and 6.7 cm.
   b. $\angle A = 45°$ and the including sides 3.8 cm. and 5.9 cm.
2. *The diagonals of a rectangle are equal.*

3. *The diagonals of a rhombus are perpendicular to each other.*

4. *The diagonals of a rhombus bisect the angles through which they pass.*

5. Show that a square is a special kind of (*a*) rectangle and (*b*) rhombus.

6. Make a list of the properties of all parallelograms.

7. Make a list of the properties of rectangles that are not true about all parallelograms.

8. Answer question 7 for (*a*) rhombuses; (*b*) squares.

B Exercises

1. *The base angles of an isosceles trapezoid are equal.*

Give two proofs. If $ABCD$ is the trapezoid, use (*a*) perpendiculars from D and C to AB; or (*b*) a segment from D parallel to CB.

2. *The diagonals of an isosceles trapezoid are equal* (Ex. 1 may be used).

3. If a diagonal of a parallelogram bisects one angle through which it passes, it bisects the other also and divides the figure into two isosceles triangles.

4. Any segment drawn through the intersection of the diagonals of a parallelogram, and terminated by the sides of the parallelogram, is bisected by the point of intersection of the diagonals.

5. The bisectors of two consecutive angles of a parallelogram are perpendicular.

6. The bisectors of two opposite angles of a parallelogram are parallel or coincide.

157. You have been studying from p. 132 to p. 135 the properties of parallelograms in general and of certain special kinds of parallelograms in particular. You are now going to find out how to prove that a given quadrilateral is a parallelogram.

THEOREM 26*

158. If a quadrilateral has one side equal and parallel to its opposite, it is a parallelogram.

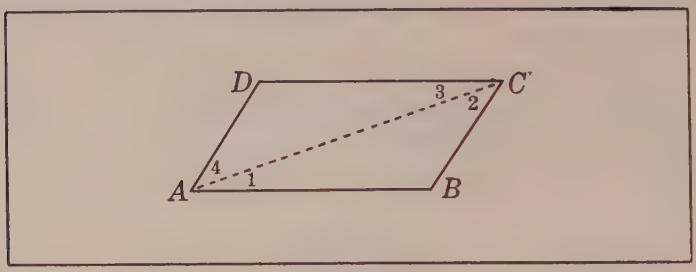


FIG. 198

Hypothesis: The quadrilateral $ABCD$ has $AB = DC$ and $AB \parallel DC$.

Conclusion: $ABCD$ is a $\square$.

Analysis and construction. Give the proof.

1. To prove $ABCD$ a $\square$, prove $AD \parallel BC$.
2. " " $AD \parallel BC$, draw AC and prove $\angle 4 = \angle 2$.
3. " " $\angle 4 = \angle 2$, prove $\triangle ADC \cong \triangle ABC$.

THEOREM 27*

159. If a quadrilateral has each side equal to its opposite, it is a parallelogram.

Hypothesis: The quadrilateral $ABCD$ has $AB = DC$ and $AD = BC$.

Conclusion: $ABCD$ is a $\square$.

Analysis and construction:

1. To prove $ABCD$ a $\square$, prove $AB \parallel DC$ and $AD \parallel BC$.
2. " " $AB \parallel DC$, " .....
3. " " $AD \parallel BC$, " .....

Complete the analysis and give the proof.

160. Methods for proving that a given quadrilateral is a parallelogram.

To prove that a quadrilateral is a parallelogram, prove either that it has

(1) Each side parallel to its opposite, or

(2) One side equal and parallel to its opposite, or

(3) Each side equal to its opposite.

C Exercises

In the following exercises consider carefully different methods of proof. The method of procedure is suggested under Ex. 1. Various possible analyses are begun. Complete each. Then select that which gives the shortest proof. Give proof in full for the one you select.

1. Given (Fig. 199) $ABCD$ a $\square$, E the mid-point of AB, and F the mid-point of DC. Prove $AECF$ a $\square$.

A. To prove $AECF$ a $\square$, prove $AF \parallel EC$ and $AE \parallel FC$.

B. To prove $AECF$ a $\square$, prove $AF \parallel EC$ and $AF = EC$.

C. To prove $AECF$ a $\square$, prove $AE \parallel FC$ and $AE = FC$.

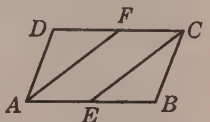


FIG. 199

D. To prove $AECF$ a $\square$, prove $AF = EC$ and $AE = FC$.

2. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

3. Given (Fig. 200) $ABCD$ a $\square$ with points E and F on the diagonal so that $AE = CF$. Prove $BFDE$ a $\square$.

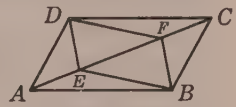


FIG. 200

4. Given $ABCD$ a $\square$. Points H and F are on AD and CB respectively so that $AH = CF$. Points E and G are on AB and CD respectively so that $AE = CG$. Prove $EFGH$ a parallelogram.

B Exercises

1. Are the diagonals of all parallelograms perpendicular to each other? of a rhombus? of a square? of a trapezoid? of a rectangle? Illustrate in each case by a figure.

2. What quadrilaterals do you know in which the diagonals are equal to each other?

3. What quadrilaterals do you know in which the angles are bisected by the diagonals?

4. The segments joining the mid-points of the sides of a parallelogram taken in order form a parallelogram.

5. Prove Ex. 3, p. 137, with points E and F on the diagonal AC extended (Fig. 200).

6. Given (Fig. 201) $ABCD$ a $\square$ with AB and CD extended so that $BE = DF$. Prove $AECF$ a $\square$.

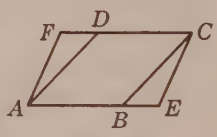


FIG. 201

7. Perpendiculars drawn to a diagonal of a parallelogram from the opposite vertices are equal.

8. In Fig. 202, $ABCD$ is a parallelogram. DX and BY are perpendicular to AC from D and B respectively. DY and BX are drawn. Prove $DXBY$ a parallelogram.

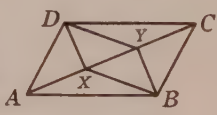


FIG. 202

9. If two sides of a quadrilateral are parallel and two opposite angles are equal, the figure is a parallelogram.

10. If one diagonal of a parallelogram bisects an angle through which it passes, the figure has all sides equal.

11. The segment joining the mid-points of two opposite sides of a parallelogram is equal and parallel to the other two sides.

12. If the diagonals of a quadrilateral bisect each other at right angles, the figure has all sides equal.

13. If the non-parallel sides of an isosceles trapezoid are extended until they intersect, two isosceles triangles are formed (see p. 135, B Ex. 1).

FIG. 203

14. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle (Fig. 203):

I. To prove $ABCD$ a $\square$, prove $\angle A =$ a rt. $\angle$.
II. " " $\angle A =$ a rt. $\angle$, " $\angle A = \angle B$.
III. " " $\angle A = \angle B$, " $\triangle ABC \cong \triangle ABD$.

For I, see definition of a rectangle; for II, apply Th. 14, p. 95.

15. If the base angles of a trapezoid are equal, the trapezoid is isosceles. Give two proofs (see p. 135, B Ex. 1).

16. Given (Fig. 204) $ABCD$ a square with AC a diagonal. If $AX = CY$, prove that $BYDX$ is a rhombus. Is this exercise true if $ABCD$ is a rhombus? if $ABCD$ is any parallelogram? Why?

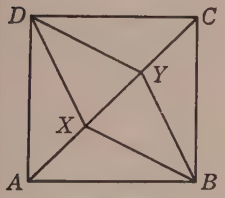
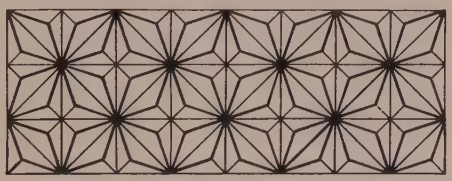


FIG. 204


From a parquet floor design
FIG. 204*a*

17. If the mid-points of the half diagonals of a parallelogram are joined in order, the figure formed is a parallelogram (see Ex. 2, p. 137).

18. Fig. 205 shows two forms of parallel rulers often used in mechanical drawing for constructing parallel lines. Show how each is constructed and upon what theorems in geometry the construction depends. Which one of these is sometimes used in folding gates? How?

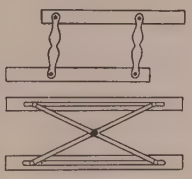


FIG. 205

19. A movable shelf is fastened to an adjustable bracket as shown in Fig. 206. Explain how the bracket is constructed and why the shelf always remains horizontal.

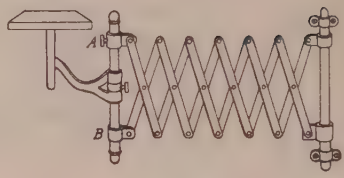


FIG. 206

NOTE. The device shown in Fig. 206 is kept from collapsing by a clamp at A. The principle of the collapse of a parallelogram is used also in hedge clippers.

A Exercises

1. The figure formed by joining the mid-points of the sides of a rhombus is a rectangle. To prove a quadrilateral a rectangle, prove that it is a parallelogram with one right angle (see definition, §146, p. 130).

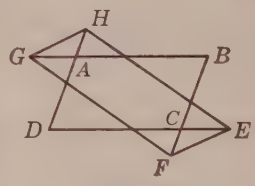


FIG. 207

2. In Fig. 207, $ABCD$ is a parallelogram. The sides are extended through A and C so that $AG = CE$ and $CF = AH$. Prove that $EFGH$ is a parallelogram.

3. In Fig. 208, $ABCD$ is a parallelogram. CD and AB are extended so that $DE = BF$, DF cuts CB at Y, and BE cuts DA at X. Prove $DX = BY$.

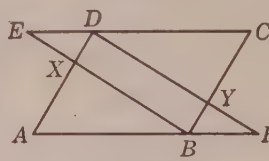


FIG. 208

4. If, in Fig. 208, DX is made equal to BY, and BX and DY are drawn and extended to meet CD at E and AB at F, respectively, prove $DE = BF$.

5. Given (Fig. 209) $\square ABCD$ with E, F, G, and H the mid-points of the sides, and the points joined as indicated in the figure. Prove that $BYDX$ is a parallelogram.

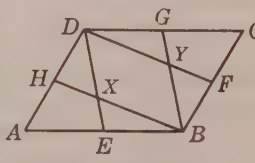


FIG. 209

6. In what other way may the points in Fig. 209 be joined so as to form a parallelogram?

7. In Fig. 210, $ABCD$ is a parallelogram. $AE = CG$ and $AH = CF$. Prove $\triangle EOH \cong \triangle GOF$.

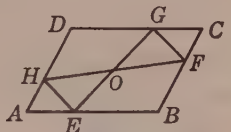


FIG. 210

NOTE. Fig. 210 is from a Roman floor design.

8. Prove Ex. 7 if E, F, G, and H are on the sides of the parallelogram extended.

9. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

10. In Fig. 211, $OXZY$ is any parallelogram, OC is any line through O. YA, XB, and ZC are drawn from Y, X, and Z respectively perpendicular to OC. Prove $OC = OA + OB$.

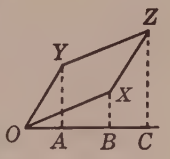


FIG. 211

11. Given (Fig. 212) $ABCD$ a square with E, F, G, and H the mid-points of the sides and the points joined as indicated. Prove $WXYZ$ a square.

To prove $WXYZ$ a square, prove it a $\square$ with $WX = XY$ and $\angle 1 =$ a rt. $\angle$.

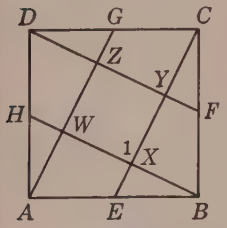


FIG. 212

From a parquet floor design

FIG. 212*a*

12. The figure formed by joining the mid-points of the sides of a rectangle in order is a rhombus.

13. The figure formed by joining the mid-points of the sides of a square in order is a square.

TOPIC 6. PARALLELS AND SEGMENTS ON TRANSVERSALS

THEOREM 28*

161. If a series of parallels cuts off equal segments on one transversal, it cuts off equal segments on all transversals.

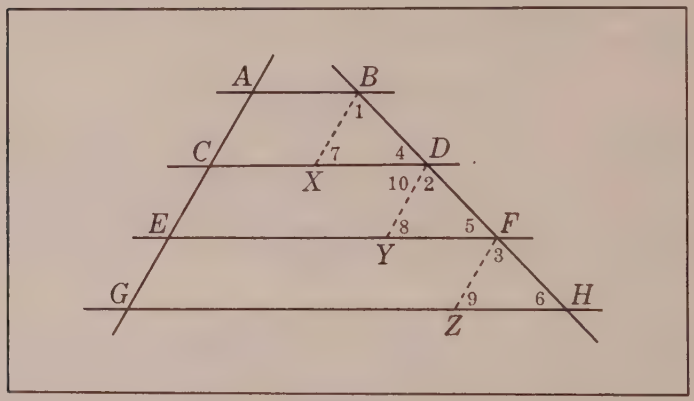


FIG. 213

Hypothesis: $AB \parallel CD \parallel EF \parallel GH$ and $AC = CE = EG$.

Conclusion: $BD = DF = FH$.

Analysis and construction:

1. To prove $BD = DF = FH$, construct BX, DY, and FZ from B, D, and F respectively and $\parallel AG$, and prove $\triangle BDX \cong \triangle DFY \cong \triangle FHZ$.
2. To prove $\triangle BDX \cong \triangle DFY$, prove $\begin{cases} \angle 1 = \angle 2, \\ \angle 7 = \angle 8, \\ BX = DY. \end{cases}$
3. " " $\angle 1 = \angle 2$, prove $BX \parallel DY$.
4. " " $\angle 7 = \angle 8$, " $\angle 7 = \angle 10$ and $\angle 8 = \angle 10$.
5. " " $BX = DY$, " $BX = AC$ and $DY = CE$.

PROBLEM 7

162. To divide a given segment into any number of equal parts.

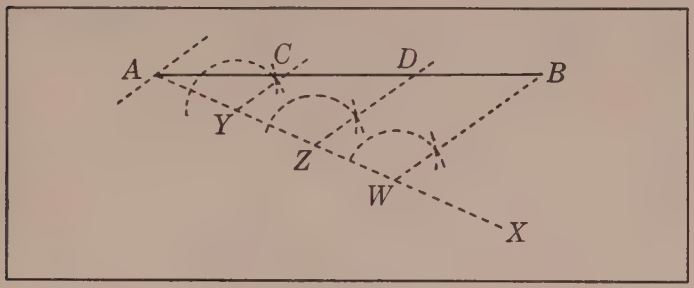


FIG. 214

Given segment AB.

To divide AB into any number of equal parts.

A. When AB is to be divided into three equal parts.

Directions:

1. From A draw ray AX making an acute angle with AB.
2. On AX from A lay off $AY = YZ = ZW$, any three equal segments.
3. Join W, the last point of division, with B.
4. From Y and Z draw parallels to WB.
5. Then AC, CD, and DB are the required parts of AB.

B. When AB is to be divided into some other number of equal parts.

Draw the figure and give the directions.

Analysis: Draw a line through $A \parallel YC$ and apply Th. 28.

163. COR. **A segment parallel to the base of a triangle and bisecting one side bisects the other side also.**

THEOREM 29 (*Optional*)

164. A segment bisecting two sides of a triangle is parallel to the third side and equal to half of it.

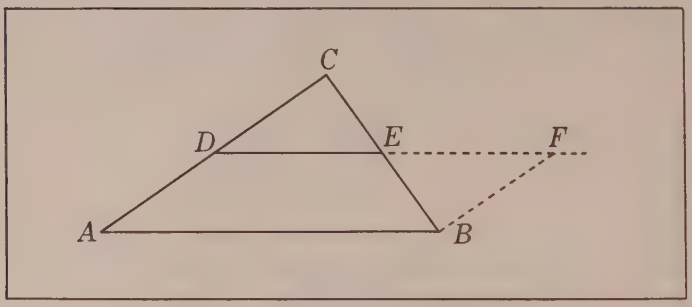


FIG. 215

Hypothesis: In $\triangle ABC$, DE joins D and E the midpoints of AC and BC respectively.

Conclusion: $DE \parallel AB$ and $DE = \frac{1}{2} AB$.

Analysis and construction:

1. To prove $DE \parallel AB$ from D, draw $BF \parallel AD$ from B, extend DE to meet BF at F, and prove $ABFD$ a $\square$.
2. To prove $ABFD$ a $\square$, prove $BF =$ and $\parallel AD$.
3. " " $BF = AD$, " $BF = DC$.
4. " " $BF = DC$, " $\triangle BFE \cong \triangle CDE$.
5. " " $DE = \frac{1}{2} AB$, " $\begin{cases} DE = \frac{1}{2} DF \text{ and} \\ DF = AB. \end{cases}$
6. " " $DF = AB$, " $ABFD$ a $\square$.

Ex. 1. Show that the corollary, §163, p. 143, is true for parallelograms and trapezoids.

Ex. 2. Prove Th. 29 true for parallelograms.

NOTE. The intersection of the medians of a triangle is called the **center of gravity** of the triangle. See Th. 30.

THEOREM 30 (*Optional*)

165. The medians of a triangle meet in a point† that is two-thirds the distance from each vertex to the mid-point of the opposite side.

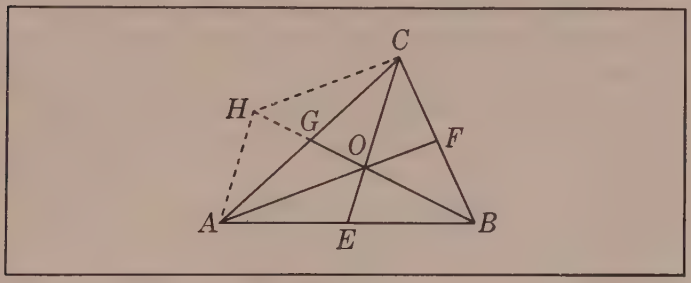


FIG. 216

Hypothesis: ABC is any triangle.

Conclusion: (1) The medians meet in a point. (2) The point of intersection is two-thirds the distance from each vertex to the mid-point of the opposite side.

Analysis and construction for (1):

1. To prove the medians meet in a point, let any two medians, as AF and CE, meet at O. Draw BO and extend to meet AC at G. Prove BG a median.
2. To prove BG a median, prove $AG = GC$.
3. " " $AG = GC$, prove that AC may be the diagonal of a parallelogram.
4. Extend BG to H so that $OH = OB$, draw HA and HC, and prove $AOCH$ a $\square$.
5. $\therefore$ prove $AH \parallel OC$ and $HC \parallel AO$.
6. $\therefore$ prove $AH \parallel EO$ and $HC \parallel OF$.

Analysis for (2): Prove $BO = \frac{2}{3}BG$; that is, $OG = \frac{1}{2}BO$.

†Lines that meet in a point are said to be **concurrent.**

B Exercises

1. Prove Th. 28, p. 142, by drawing the parallels from A, C, and E parallel to BH. Use the same hypothesis and conclusion as is given for Th. 28.

2. A line may be divided into any number of equal parts (for example, 5) by the construction shown in Fig. 217. Give the complete directions and the proof.

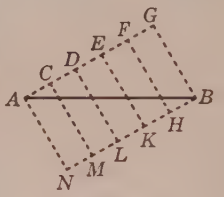


FIG. 217

NOTE. A sheet of ruled paper may be used to divide a segment into a given number of equal parts. Number the lines as in Fig. 218. If the segment is to be divided into 7 equal parts, put the ends of the segment on lines 0 and 7. Why?

3. Show how a carpenter's steel square may be used to divide a board into strips of equal width (Fig. 219).

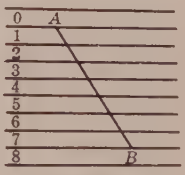


FIG. 218

NOTE. By the *width* of the board is meant the perpendicular distance between the sides.

4. The segments which join the mid-points of the sides of a quadrilateral taken in order form a parallelogram (Fig. 220).

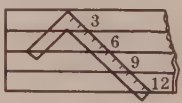


FIG. 219

5. Prove Ex. 4 for a concave quadrilateral and for a cross quadrilateral.

6. The medians of a quadrilateral bisect each other.

The segment joining the mid-points of two opposite sides of a quadrilateral is called a **median** of the quadrilateral.

7. The segments which join the mid-points of two opposite sides of a quadrilateral to the mid-points of the diagonals form a parallelogram.

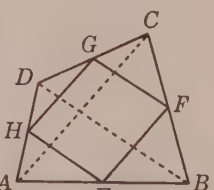


FIG. 220

A Exercises

1. Perpendiculars from the mid-points of two sides of a triangle to the third side are equal. How might this exercise be proved if the given triangle were isosceles?

2. Through a given point within an angle construct a segment terminated by the sides of the angle and bisected by the given point.

3. The median from the vertex of the right angle of a right triangle to the hypotenuse is one-half the hypotenuse.

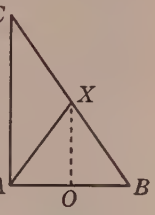


FIG. 221

Suggestion. Draw XO (Fig. 221) from $X \parallel CA$ and prove XO the perpendicular bisector of AB.

4. If from two opposite vertices of a parallelogram segments are drawn to the mid-points of the opposite sides, these segments trisect the diagonal joining the other two vertices.

5. It is said that an Arab, about 900 A.D., trisected a given segment as in Fig. 222. Show that this is the same construction as that given in Ex. 4.

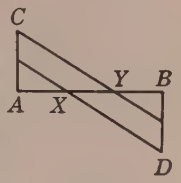


FIG. 222

6. A segment joining the mid-points of the non-parallel sides of a trapezoid bisects both diagonals.

7. Segments drawn from one vertex of a parallelogram to the mid-points of the opposite sides trisect the diagonal which they intersect (Fig. 223).

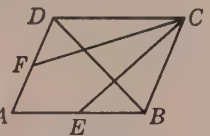


FIG. 223

Suggestion. Draw diagonal AC and apply Th. 30.

8. A segment joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and equal to half their sum.

Survey Test

1. Define a quadrilateral.

2. (*a*) What is a trapezoid? an isosceles trapezoid? (*b*) What are the non-parallel sides of a trapezoid called? the parallel sides? (*c*) How many bases has a trapezoid?

3. (*a*) Define a parallelogram. (*b*) Name the opposite and consecutive sides of a parallelogram lettered *ABCD*; the opposite angles; the consecutive angles. (*c*) Which are usually considered the bases of a parallelogram? (*d*) Define the altitude of a parallelogram. (*e*) What are the diagonals? (*f*) What can you say of the diagonals of a parallelogram in respect to each other?

4. (*a*) Name the properties of parallelograms. (*b*) Which one is stated in the definition of parallelograms?

5. What property or properties of parallelograms could be used to prove segments equal? angles equal? triangles congruent?

6. Quote two corollaries that depend upon the fact that the opposite sides of a parallelogram are equal.

7. (*a*) Define a rectangle. (*b*) Name the properties of a rectangle that are not properties of all parallelograms.

8. (*a*) Define a rhombus. (*b*) Name the properties of a rhombus that are not properties of all parallelograms.

9. Define a square.

10. Quote in full the theorems by which you can prove that a given quadrilateral is a parallelogram.

11. If a series of parallels cuts off equal segments on one transversal, what is true of the segments cut on all transversals by the given series of parallel lines?

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	129	5	132, 134, 133	9	130
2	129	6	132	10	136, 137
3	129, 134	7	130, 133, 134	11	142
4	129, 132, 133, 134	8	130, 133, 135		

166. SUMMARY I

Important Points in Units I and II, Topics 1-6

Any two triangles are congruent if:

1. Two sides and the included angle.....Th. 1, p. 47
2. Two angles and the included side.....Th. 2, p. 49
3. Three sides.....Th. 4, p. 58

Two right triangles are congruent if:

1. The hypotenuse and an acute angle.....Th. 22, p. 121
2. The hypotenuse and a side.....Th. 23, p. 122

A line is located definitely if it:

1. Contains two given points.....Post. 2, p. 19
2. Bisects a given angle.....Post. 9, p. 31
3. Is perpendicular to a line through a given point.....Posts. 11 and 12, p. 31
4. Is parallel to a line through a point.....Post. 22, p. 84

Equal segments are formed by:

1. Segments equal to the same or to equal segments Ax. 5, p. 30
2. Sums, differences, equal multiples, or equal parts of equal segments.....Axs. 1–4, pp. 29–30
3. Radii of the same or equal circles.....Post. 21, p. 45
4. Sides of an isosceles triangle.....§60, p. 44
5. Corresponding sides of congruent triangles.....§76, p. 52
6. Opposite sides of a parallelogram.....§149, p. 132
7. Parallel or perpendicular segments between parallels.....§§150–151, p. 132
8. Sides of a rhombus or square.....§152, p. 133
9. The intersection of the diagonals of a parallelogram Th. 25, p. 134
10. The diagonals of a rectangle.....Ex. 2, p. 134

Equal segments are formed by:

11. The non-parallel sides of an isosceles trapezoid.... §143, p. 129
12. A series of parallels cutting equal segments on one transversal and intersecting a second.....Th. 28, p. 142
13. A segment parallel to the base of a triangle and bisecting one side.....§163, p. 143

Two angles are equal if they are:

1. Equal to the same or to equal angles.....Ax. 5, p. 30
2. Sums, differences, equal multiples, or equal parts of equal angles.....Axs. 1–4, pp. 29–30
3. Right angles or straight angles.....Posts. 13, 14, p. 31
4. Supplements or complements of the same or of equal angles.....Posts. 15, 16, p. 31
5. Vertical angles.....Post. 17, p. 31
6. Corresponding angles of congruent triangles.....§76, p. 52
7. Base angles of an isosceles triangle.....Th. 3, p. 57
8. Alternate interior angles of parallel lines.....Th. 12, p. 93
9. Corresponding angles of parallel lines.....Th. 13, p. 94
10. The third angles of two triangles that have two angles of one equal respectively to two angles of the other..§122, p. 104
11. Angles with their sides parallel or perpendicular right side to right side.....Th. 16, p. 99; Th. 19,‡ p. 110
12. Opposite angles of a parallelogram.....§153, p. 133

‡13. Base angles of an isosceles trapezoid.....B Ex. 1, p. 135

Sums of angles:

1. The sum of the interior angles of a triangle.....Th. 17, p. 103
2. The exterior angle of a triangle equals the sum of the two non-adjacent interior angles.....Th. 18, p. 106
3. The sum of the interior angles of any polygon.....Th. 20, p. 112
4. The sum of the exterior angles of any polygon.....Ex. 6, p. 114

‡ Optional.

Angles are supplementary if:

1. Their sum is a straight angle ..... §45, p. 23
2. They are interior angles on the same side of a transversal cutting two parallels ..... Th. 14, p. 95
3. They have their sides parallel or perpendicular right side to left side ..... Th. 16, p. 99; Th. 19,‡ p. 110

Lines are perpendicular if:

1. One line passes through a point in another so that the adjacent angles formed are equal ..... §36, p. 22
2. One is perpendicular to a line parallel to the other ... Th. 15, p. 96
3. One of them passes through two points each equally distant from the ends of a segment of the other ..... Th. 5, p. 73

Two straight lines in the same plane are parallel if:

1. The alternate interior angles are equal ..... Th. 8, p. 86
2. The corresponding angles are equal ..... Th. 9, p. 87
3. The interior angles on the same side of the transversal are supplementary ..... Th. 10, p. 88
4. They are parallel to the same line ..... Th. 7, p. 85
5. They are perpendicular to the same line ..... Th. 11, p. 89
6. One is one side of a triangle and the other passes through the mid-points of the other two sides ..... Th. 29,‡ p. 144

Construction problems:

Construct:

1. An angle equal to a given angle ..... p. 61
2. The bisector of a given angle ..... p. 62
3. A perpendicular to a given line ..... pp. 64, 74
4. The perpendicular bisector of a given segment ..... p. 75
5. A parallel to a given line ..... p. 90

To divide a segment into any number of equal parts ..... p. 143

‡ Optional

Isosceles triangles:

A triangle is isosceles if:

1. At least two sides are equal ..... §60, p. 44
2. At least two angles are equal ..... Th. 21, p. 119

Properties of:

1. Two sides are equal ..... §60, p. 44
2. The base angles are equal ..... Th. 3, p. 57
3. The altitude bisects the base and the vertex angle ... §140, p. 123

Isosceles trapezoids:

A trapezoid is isosceles if its non-parallel sides are equal .. §143, p. 129

Properties of:

1. Two sides are parallel ..... §143, p. 129
2. The non-parallel sides are equal ..... §143, p. 129
‡3. The base angles are equal ..... B Ex. 1, p. 135
‡4. The diagonals are equal ..... B Ex. 2, p. 135

Parallelograms:

A quadrilateral is a parallelogram if:

1. Each side is parallel to its opposite ..... §144, p. 129
2. One side is equal and parallel to its opposite ..... Th. 26, p. 136
3. Each side is equal to its opposite ..... Th. 27, p. 136
4. The diagonals bisect each other ..... Ex. 2, p. 137

Properties of:

1. The opposite sides are parallel ..... §144, p. 129
2. The opposite sides are equal ..... §149, p. 132
3. The opposite angles are equal ..... §153, p. 133
4. The diagonal bisects the parallelogram ..... Th. 24, p. 132
5. The diagonals bisect each other ..... Th. 25, p. 134

Rhombuses:

A quadrilateral is a rhombus if it is a parallelogram with two consecutive sides equal ..... §146, p. 130

‡ Optional.

Properties of:

1. All properties of parallelograms.
2. All sides are equal ..... §152, p. 133
3. The diagonals are perpendicular to each other .... C Ex. 3, p. 135
4. The diagonals bisect the angles ..... C Ex. 4, p. 135

Rectangles:

A quadrilateral is a rectangle if it is a parallelogram with one right angle ..... §146, p. 130

Properties of:

1. All properties of parallelograms.
2. All angles are right angles ..... §154, p. 133
3. The diagonals are equal ..... Ex. 2, p. 134

Squares:

A quadrilateral is a square if it is both a rectangle and a rhombus ..... §146, p. 130

Properties of:

All properties of both rectangles and rhombuses.

C Exercises — Oral Review

1. Do the lengths of the sides prove anything about the size of an angle? Why?

2. Through how many degrees does the hour hand of a clock revolve in five hours?

3. When is an angle less than its supplement? equal to its supplement? greater than its supplement?

4. What is the complement of 90°? of 0°? What is the supplement of 180°?

5. Find the supplement of an angle if its complement is 15°?

6. Are two angles with a common vertex and a common side adjacent? Why?

7. What is the hypotenuse of a right triangle? What are the other two sides called?

8. Why may an equilateral triangle be considered isosceles?

9. Is a triangle a polygon? Why?

10. What is the sum of the acute angles of a right triangle? Why?

Complete the three following statements and quote the theorem for each in full:

11. If line $x \perp$ line a and line $y \perp$ line a, then .....

12. If line $x \parallel$ line y and line $a \perp$ line x, then .....

13. If line $x \parallel$ line a and line $y \parallel$ line a, then .....

14. If a theorem has been proved, why cannot the converse be accepted as true without proof?

15. If two angles of a triangle are 70° and 85°, what is the third angle?

16. If the vertex angle of an isosceles triangle is 72°, how many degrees are there in each base angle? Why?

17. If one base angle of an isosceles triangle is 40°, how many degrees in the vertex angle? Why?

18. If one base angle of an isosceles triangle is 50°, how many degrees in the exterior angle at the vertex? Why?

19. How do you get the proof of a theorem from the analysis?

20. How is an angle of 60° obtained? an angle of 45°?

21. What is an altitude of a triangle? a median? In what kind of triangle does the altitude bisect the base?

22. Draw a triangle that has an altitude that falls outside the triangle. How many altitudes has a triangle? Why?

23. Name three ways of locating a line that will divide an isosceles triangle into two congruent triangles.

24. Give five methods of proving lines parallel. Which of these were proved by means of the definition of parallel lines? On what was the proof of the others based?

25. State two methods of proving a triangle isosceles. Which of these depends directly upon the definition of an isosceles triangle?

26. What is an indirect proof? Describe it fully.

27. What theorems did you prove by means of an indirect proof?

28. (*a*) What is the difference between a rhombus and a square? (*b*) What is the difference between a square and a rectangle? (*c*) Why is an isosceles trapezoid called isosceles?

29. Give three methods that may be used to prove that a given 4-side is a parallelogram. Which of these is obtained from the definition of a parallelogram?

30. What is meant by the distance from a point to a line? by the distance between two points? by the distance between two parallels?

31. What is meant by the altitude of a parallelogram? How many different altitudes has a parallelogram? Why?

32. Answer question 31 for any trapezoid.

33. What illustrations of figures and theorems that you have had can you find in Fig. 241, p. 160?

B Exercises — Miscellaneous Review

NOTE TO THE TEACHER. The following exercises are suitable for rapid review work. They may be used in whole or in part, when the class reaches this point in the work, or from time to time as the work progresses. Work marked optional is in no way involved.

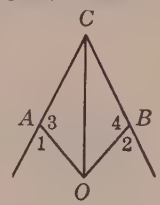


FIG. 224

1. If, in Fig. 224, $AC = BC$ and $AO = BO$, what theorem would prove $\triangle ACO \cong \triangle BCO$?

2. If, in Fig. 224, $AC = BC$ and $AO = BO$, make an analysis to prove $\angle 1 = \angle 2$.

3. In using the analysis made in answer to Ex. 2, what would be the final reason used to prove $\angle 1 = \angle 2$?

4. If, in Fig. 225, $CA = CB$, $\angle 1 = \angle 2$, and $AX = BY$, name the equal parts that could be used to prove $\triangle ACX \cong \triangle BCY$.

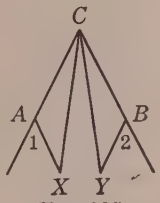


FIG. 225

5. Using the data given in Ex. 4, prove infull that $CX = CY$.

6. Two boys wish to find the distance across a pond from A to C (Fig. 226). Show from the figure how this may be done if they have some means of constructing an angle equal to a given angle.

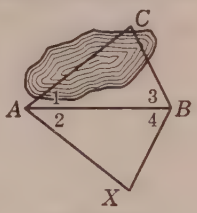


FIG. 226

7. If, in Fig. 227, $CA = CB$ and $\angle 1 = \angle 2$, name the equal parts that could be used to prove $\triangle ACY \cong \triangle BCX$.

8. If, in Fig. 227, $\triangle ACY$ had been proved congruent to $\triangle BCX$ as in Ex. 7, what reason could be used to prove $AX = BY$?

9. If, in Fig. 227, $CA = CB$ and $CX = CY$, what triangles could be used to prove $\angle 3 = \angle 4$? Give proof.

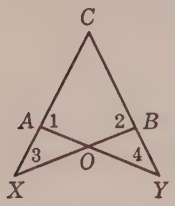


FIG. 227

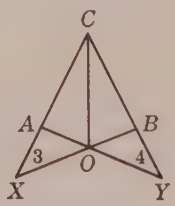


FIG. 228

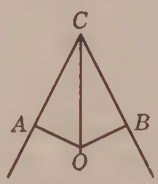


FIG. 229

10. In Fig. 228, $CA = CB$, CO bisects $\angle C$, O is any point in CO, and the points are joined as indicated. Name the equal parts that could be used to prove $\triangle COA \cong \triangle COB$.

11. Supposing that $\triangle COA$ had been proved congruent to $\triangle COB$ as in Ex. 10 (Fig. 228), name the equal parts that could be used to prove $\triangle CAY \cong \triangle CBX$.

12. Using the data given in Ex. 10, make a complete analysis to prove $\angle 3 = \angle 4$ (Fig. 228).

13. Given (Fig. 229) $CA = CB$, $OA \perp CA$, and $OB \perp CB$. Prove $\triangle COA \cong \triangle COB$ by congruent right triangles.

14. Given (Fig. 229) CO bisecting $\angle C$. From O, any point in CO, OA is drawn $\perp CA$ and $OB \perp CB$. Prove $\triangle COA \cong \triangle COB$ by congruent right triangles.

15. If, in Fig. 230, $\angle 1 = \angle 2$, prove $\angle 3 = \angle 4$.

16. If, in Fig. 230, $\angle 2 = \angle 5$, prove $\angle 4 = \angle 6$.

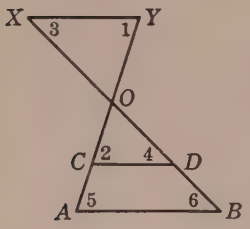


FIG. 230

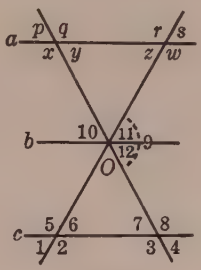


FIG. 231

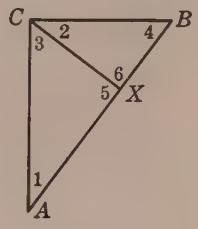


FIG. 232

Use Fig. 231 for Exs. 17–26. $a \parallel b \parallel c$ and $\angle 6 = \angle 7$. Prove:

17. $\angle y = \angle z$.

18. $\angle s = \angle 7$.

19. $\angle 2 = \angle r$.

20. $\angle w + \angle 7 = 2$ rt. $\angle$s.

21. $\angle 9 = \angle s + \angle 4$.

22. $\angle p = \angle 1$.

23. $\angle x = \angle 5$.

24. $\angle q = \angle r$.

25. $\angle q + \angle 1 = 2$ rt. $\angle$s.

26. $\angle 10 = \angle 11$.

27. In Fig. 232, $AC \perp CB$ and $CX \perp AB$. If $\angle 1 = 26°$, find the number of degrees in $\angle 1$, $\angle 3$, and $\angle 4$. Prove without the use of degrees that $\angle 1 = \angle 2$.

28. Given (Fig. 233) $a \parallel b$, cut by line c. XY and ZW both $\perp c$ at points Y and W respectively. Prove $\angle 1 = \angle 2$.

29. Given (Fig. 233) $a \parallel b$, cut by line c, and $\angle 1 = \angle 2$. Prove $XY \parallel ZW$.

30. The bisectors of a pair of alternate interior angles of parallel lines are parallel.

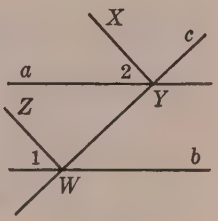


FIG. 233

31. If, in Fig. 234, $a \parallel b$ and DF bisects $\angle 1$, prove $\triangle CDF$ isosceles.

32. If (Fig. 234) $a \parallel b$, DE bisects $\angle 1$, and CE bisects $\angle 2$, prove $CE \perp DF$.

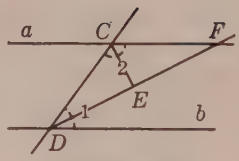


FIG. 234

33. Test the accuracy of the right angle of your triangle thus: place your triangle on your T square in position ABC (Fig. 235) and draw the perpendicular AB. Turn the triangle over to position ABC'. The perpendicular should coincide with the edge AB when the triangle is in position ABC'. Why?

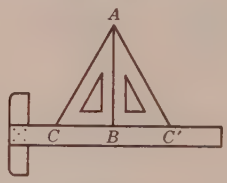


FIG. 235

34. If, in Fig. 236, $ABCD$ is a parallelogram, $AE = CG$, and $\angle 1 = \angle 2$, prove $AH = CF$.

35. What property of parallelograms must be used in answering Ex. 34?

36. Using the data given in Ex. 34, make an analysis to prove $EFGH$ a parallelogram. Give proof.

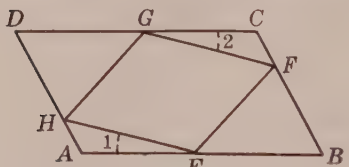


FIG. 236

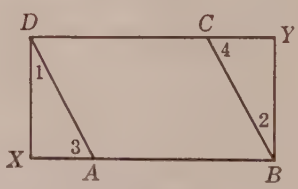


FIG. 237

Use Fig. 237 for Exs. 37–39. $ABCD$ is a parallelogram. BA and DC are extended.

37. Prove $\angle 3 = \angle 4$.

38. If $DX \perp BX$ and $BY \perp DY$, prove $DX = BY$.

39. If $\angle 1 = \angle 2$, prove $DX = BY$ and $DX \parallel BY$.

40. If, in Fig. 237, $BYDX$ is a rectangle and $\angle 1 = \angle 2$, prove $ABCD$ a parallelogram.

Use Fig. 238 for Exs. 41–43. $ABCD$ is a parallelogram and $\angle 1 = \angle 2$.

41. Prove $\triangle ADX \cong \triangle CBY$ and $DX \parallel BY$.

42. What property of parallelograms would be used in answering Ex. 41?

43. Make a complete analysis to prove $DXBY$ a parallelogram. Give proof in full.

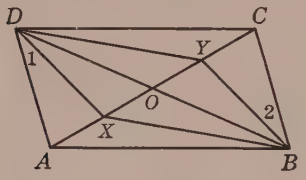


FIG. 238

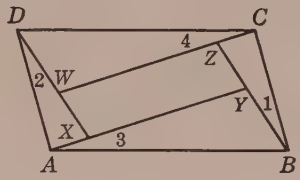


FIG. 239

44. In Fig. 239, $WXYZ$ is a parallelogram. If XW and ZY are extended so that $WD = YB$ and YX and WZ are extended so that $XA = ZC$, prove $ABCD$ a parallelogram.

45. In Fig. 239, $ABCD$ is a parallelogram. If $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, prove $WXYZ$ a parallelogram.

46. Show that the distance between two inaccessible points, as A and B on opposite sides of a stream, may be found as follows (Fig. 240):

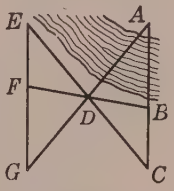


FIG. 240

Set a stake at C, sighting it in line with AB. BC may be any convenient distance. Take D any point from which A, B, and C are visible. Sight E in line with D and C, making $ED = DC$. Sight F in line with D and B, making $FD = DB$. Sight G so that it will be in line with F and E and also in line with D and A. What line should be measured to find the distance AB?

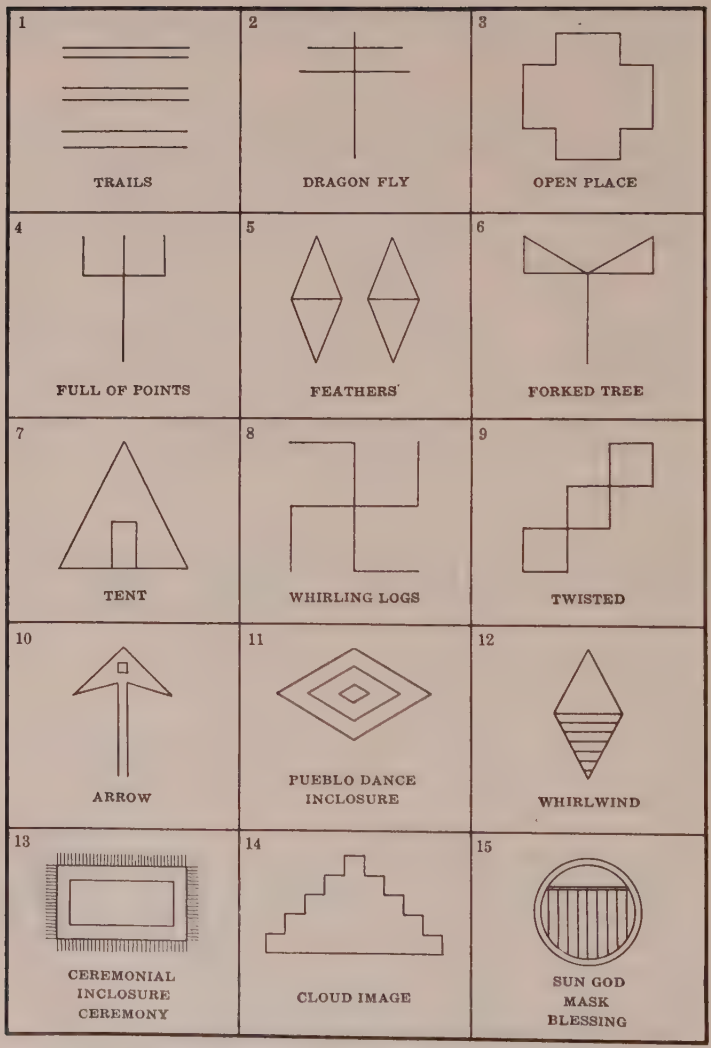


FIG. 241. **Some design motifs used by American Indians with their meanings**

167. The 60°–30° right triangle (*optional*).

NOTE TO THE TEACHER. As the 60°–30° right triangle is used in many practical applications of numerical trigonometry (see pages 301–307), the two following exercises are inserted, to be used at your option.

Ex. 1. In a right triangle whose acute angles are 60° and 30° respectively, the side opposite the smaller angle is half the hypotenuse.

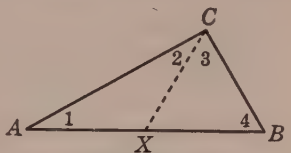


FIG. 242

Analysis (Fig. 242):

I. To prove $CB = \frac{1}{2}AB$, draw CX so that $\angle 3 = \angle 4$; prove $BC = BX = CX = AX$.

II. Since $\angle 3 = \angle 4$, $\triangle CXB$ is isosceles and $BX = CX$.

III. To prove $CX = AX$, prove $\angle 1 = \angle 2 = 30°$.

IV. " " $BC = BX$, prove $\angle CXB = 60°$.

Ex. 2. If one leg of a right triangle is half the hypotenuse, the angle opposite this leg is 30°.

Analysis (Fig. 242):

I. To prove $\angle 1 = 30°$, prove $\angle 4 = 60°$.

II. $\therefore$ draw CX so that $\angle 3 = \angle 4$ and prove $\triangle CXB$ equilateral.

Outline of proof. Give the reasons.

1. $\angle 3 = \angle 4$.
2. $\therefore CX = BX$.
3. $\angle 1$ is complement of $\angle 4$.
4. $\angle 2$ " " " $\angle 3$.
5. $\therefore \angle 1 = \angle 2$.
6. $\therefore AX = CX$.
7. $\therefore BX = CX = AX$.
8. But $BC = BX$.
9. $\therefore \triangle BCX$ is equilateral.
10. $\therefore \angle 4 = 60°$.
11. $\therefore \angle 1 = 30°$.

TOPIC 7. INEQUALITIES (*Optional*)†

168. Statements of inequality. Statements like $a=b$ and $\angle A=\angle B$ are called **statements of equality.** Statements like $a>b$ and $x<y$ are called **statements of inequality,** or simply **inequalities.** They are important in both algebra and geometry.

169. For the meaning of $>$ and $<$ see page 14. Notice that the sign always opens toward the greater: thus $4>2$ or $2<4$. $a>b$ means the same thing as $b<a$.

170. Two inequalities are said to be stated in the **same order** when the signs point in the same direction; thus $a>b$ and $c>d$. They are said to be stated in the **opposite order** when the signs point in the opposite direction: thus $a>b$ and $d<c$.

171. The following general assumptions or axioms tell you how you may work with inequalities:

Ax. 9. If equals are added to unequals, the results are unequal in the same order.

Thus: If $x>y$, then $x+2>y+2$.
If $\angle A<\angle B$, then $\angle A+\angle C<\angle B+\angle C$.

Ax. 10. If equals are subtracted from unequals, the results are unequal in the same order.

Thus: If $x<y$, then $x-4<y-4$.
If $a>b$, then $a-c>b-c$.

Ax. 11. If unequals are added to unequals, the greater to the greater and the lesser to the lesser, the results are unequal in the same order.

Thus: If $a>b$ and $c>d$, then $a+c>b+d$.
If $x<y$, then $x+4<y+5$.

†*To the teacher:* Theorems 31 and 32 in this topic must not be omitted if Prob. 9 of Topic 9, p. 174, is to be studied. The entire topic is a prerequisite for inequalities in circles and for work in trigonometry and solid geometry.

Ax. 12. If unequals are subtracted from equals, the results are unequal in the opposite order.

Thus: If $a>b$, then $x-a<x-b$.
If $d<c$, then $10-d>10-c$.

Ax. 13. If unequals are multiplied by the same positive number, the results are unequal in the same order.

Thus: If $\angle A>\angle B$, then $2\angle A>2\angle B$.
If $a<b$, then $2a<2b$.

Ax. 14. If unequals are divided by the same positive number, the results are unequal in the same order.

Thus: If $\angle A<(\angle B+\angle C)$, then $\frac{1}{2}\angle A<\frac{1}{2}(\angle B+\angle C)$.
If $a>b$, then $\frac{1}{2}a>\frac{1}{2}b$.

172. The following are of fundamental importance in inequalities in geometry:

I. For angles:

THEOREM 6. **An exterior angle of a triangle is greater than either of the non-adjacent interior angles** (p. 83).

II. For segments:

Post. 23. The sum of two sides of a triangle is greater than the third.

This postulate is known to everyone who goes *'cross lots* instead of *around the corner.*

Post. 24. The difference between two sides of a triangle is less than the third.

III. In general:

Ax. 7. The whole is greater than any of its parts (p. 82).

Ax. 8. If the first quantity is greater than the second and the second is greater than or equal to the third, then the first is greater than the third (see p. 82).

Ex. 1. How many triangles can be formed of the sides 7, 11, 19, and 15 cm.?

Ex. 2. If two sides of a triangle are 8 cm. and 15 cm., what are the upper and lower limits of the third side?

THEOREM 31

173. If one angle of a triangle is greater than a second, the side opposite the greater angle is greater than the side opposite the lesser angle.

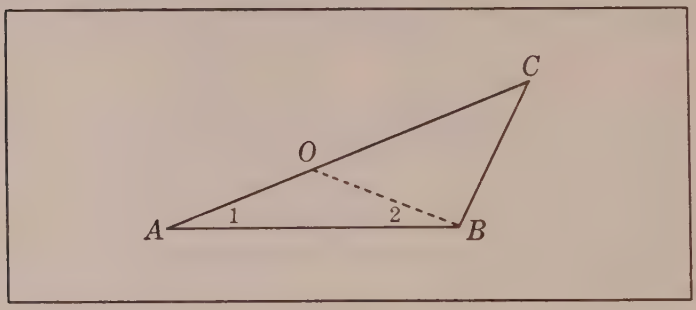


FIG. 243

Hypothesis: In $\triangle ABC$, $\angle B > \angle A$.

Conclusion: $AC > CB$.

Analysis and construction:

1. To prove $AC > CB$, apply Post. 23, p. 163.
2. Construct a triangle with BC for one side and the sum of the other two sides equal to AC.
3. $\therefore$ construct OB so that $\angle 2 = \angle 1$ and prove that $AC = OB + OC$.
4. $\therefore$ prove $OA = OB$.

Outline of proof. Give the reasons.

1. *a.* $AO = OB$.
   b. $AO + OC = OB + OC$.
   c. $\therefore AC = OB + OC$.
2. $OB + OC > CB$.
3. $\therefore AC > CB$.

174. COR. **The perpendicular is the shortest segment from a point to a straight line.**

THEOREM 32

175. If one side of a triangle is greater than a second, the angle opposite the greater side is greater than the angle opposite the lesser side.

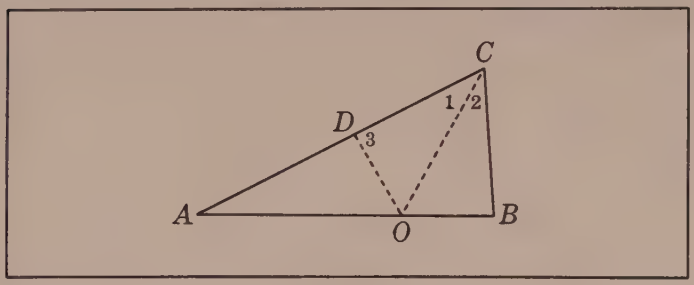


FIG. 244

Hypothesis: In $\triangle ABC$, $AC > CB$.

Conclusion: $\angle B > \angle A$.

Analysis and construction:

1. To prove $\angle B > \angle A$, prove $\angle B$ equal to an angle that is greater than $\angle A$ (use Th. 6, p. 83).
2. $\therefore$ draw CO bisecting $\angle C$ and meeting AB at O. On CA lay off $CD = CB$. Draw DO and prove $\angle B = \angle 3$ and $\angle 3 > \angle A$.
3. To prove $\angle B = \angle 3$, prove $\triangle COB \cong \triangle COD$.

Outline of proof. Supply missing statements and give reasons.

1. *a.* $\triangle COB \cong \triangle COD$.
   b. $\therefore \angle B = \angle 3$.
2. *a.* $\angle 3 > \angle A$.
   b. $\therefore \angle B > \angle A$.

Exercise. Prove Th. 32 by the following method: On CA lay off $CD = CB$, draw DB, and prove $\angle B > \angle 1$, $\angle 1 = \angle 2$, $\angle 2 > \angle 3$ (Fig. 245).

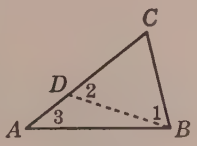


FIG. 245

THEOREM 33

176. If two triangles have two sides of one equal to two sides of the other but the included angles unequal, the triangle that has the greater included angle has the greater third side.

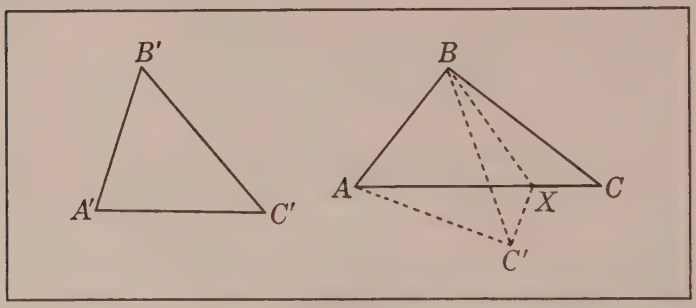


FIG. 246

Hypothesis: $\triangle ABC$ and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$, and $\angle B > \angle B'$.

Conclusion: $AC > A'C'$.

Analysis and construction:

1. To prove $AC > A'C'$, apply Post. 23, p. 163.
2. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with AB, A' on A and B' on B. Bisect $\angle C'BC$; let the bisector meet AC at X; draw XC'. Prove $AX + XC' = AC$ and $> AC'$.
3. To prove $AX + XC' = AC$, prove $XC = XC'$.
4. " " $XC = XC'$, " $\triangle CBX \cong \triangle C'BX$.

Discussion: In constructing Fig. 246, point C' might fall on side AC or within $\triangle ABC$. Give the proof in each case.

THEOREM 34

177. If two triangles have two sides of one equal to two sides of the other but the third sides unequal, the triangle that has the greater third side has the greater angle opposite.

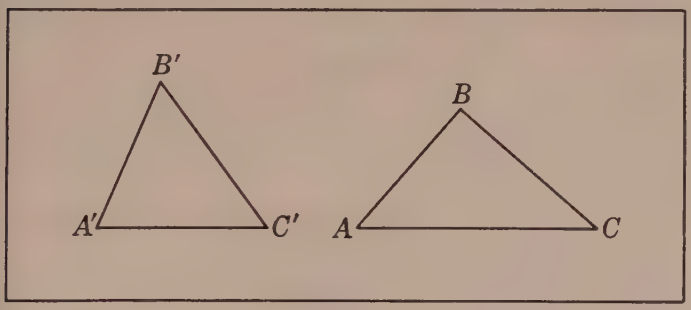


FIG. 247

Hypothesis: $\triangle ABC$ and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$, and $AC > A'C'$.

Conclusion: $\angle B > \angle B'$.

Analysis: To prove $\angle B > \angle B'$, prove that $\angle B$ cannot be equal to or less than $\angle B'$.

Outline of proof. Give the reasons.

1. If $\angle B = \angle B'$,
   - *a.* $\triangle ABC$ would $\cong \triangle A'B'C'$
   - *b.* and AC would $= A'C'$.
   - *c.* But $AC > A'C'$.
   - *d.* $\therefore \angle B$ is not $= \angle B'$.
2. If $\angle B < \angle B'$,
   - *a.* then AC would $< A'C'$.
   - *b.* But $AC > A'C'$.
   - *c.* $\therefore \angle B$ is not $< \angle B'$.
3. $\therefore \angle B$ must be $> \angle B'$.

A Exercises

1. If from a point within a triangle segments are drawn to the vertices of the triangle, the sum of these segments is greater than one-half the perimeter of the triangle.

2. If, in $\square ABCD$, $AB > AD$, the diagonal DB does not bisect $\angle D$, but $\angle ADB > \angle BDC$.

3. If, in $\triangle ABC$, the median AD is drawn and $\angle ADB$ is acute, $AC > AB$.

4. If, in $\square ABCD$, $\angle A < \angle B$, the diagonal opposite $\angle A$ is less than the diagonal opposite $\angle B$.

5. If from a point within a triangle segments are drawn to the extremities of one side, their sum is less than the sum of the other two sides of the triangle.

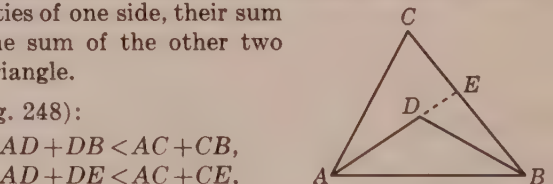


FIG. 248

Analysis (Fig. 248):

To prove $AD + DB < AC + CB$,
" $AD + DE < AC + CE$,
and $DB < DE + EB$,
and add.

6. If from a point in a perpendicular to a straight line two oblique segments are drawn cutting the straight line at unequal distances from the foot of the perpendicular, the more remote is the greater (Fig. 249).

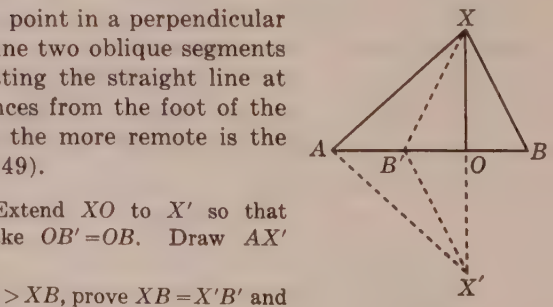


FIG. 249

Suggestion. Extend XO to X' so that $OX' = OX$. Make $OB' = OB$. Draw AX' and $B'X'$.

To prove $XA > XB$, prove $XB = X'B'$ and $AX = AX'$ and apply Ex. 5.

7. State and prove the converse to Ex. 6. Use an indirect proof like that used for Th. 34.

TOPIC 8. CONSTRUCTION OF STRAIGHT-LINE FIGURES *(Optional)*

178. The construction of many straight-line figures depends directly upon the construction of triangles. Before attempting the work that follows, review these exercises.

A. Construct a triangle given:
   (1) Two angles and the included side.
   (2) Two sides and the included angle.
   (3) Three sides.

B. Construct a right triangle given:
   (1) The hypotenuse and an acute angle.
   (2) The hypotenuse and a side.

179. The **parts of a triangle** will be indicated as follows:

a. A, B, and C refer to the vertices. In a right triangle, C refers to the vertex of the right angle.

b. a, b, and c refer to the sides opposite $\measuredangle A$, B, and C respectively. In a right triangle, c refers to the hypotenuse.

c. h_a refers to the altitude from the vertex A to side a. h_b and h_c have similar meanings.

d. m_a refers to the median from the vertex A to the mid-point of side a. m_b and m_c have similar meanings.

e. t_a refers to the bisector of $\angle A$ from the vertex of the angle to its intersection with side a. t_b and t_c have similar meanings.

Ex. 1. Draw a triangle not a right triangle. Letter sides and angles as stated above. Draw h_c, m_a, and t_b and letter.

Ex. 2. Work Ex. 1 for a right triangle.

B Exercises

The method of studying many construction problems is indicated in the suggestions given under Exs. 1 and 2. Note the discussion called for under Ex. 1. Such a discussion should be given for all exercises and should answer the questions there given.

1. Construct $\triangle ABC$ given a, b, *and* h_c.

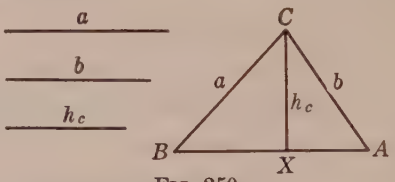


FIG. 250

Analysis. Draw any $\triangle ABC$ (Fig. 250) with the altitude on side c. Mark a, b, and h_c on the figure as shown. Note that $\triangle ABC$ consists of two triangles, each of which comes under §178, B (2).

Directions. To construct the triangle so that a, b, and h_c come equal to the predetermined lengths given, (1) draw AB any length; (2) draw a perpendicular to AB at X any point in AB and make it equal to h_c. Complete the figure.

Discussion. Would any change in the data given make the problem impossible? If so, what? How many triangles can be constructed using the data given? Show how.

2. Construct $\triangle ABC$ with $c = 10$ cm., $b = 7$ cm., $m_c = 4$ cm.

Suggestion. Draw any $\triangle ABC$ with the median CX to side c (Fig. 251). Mark $b = 7$, $AX = 5$, $BX = 5$, and $CX = 4$. Show how to construct $\triangle AXC$ and BXC. Construct the triangle with b, c, and m_c the required lengths.

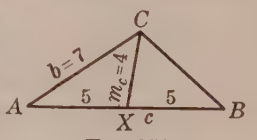


FIG. 251

In the discussion note that the data given cannot be changed. Could more than one triangle be constructed? Why?

Construct $\triangle ABC$ given:

3. $\angle A$, b, t_a.

4. a, h_a, m_a.

5. h_b, m_b, a.

6. a, h_b, m_a.

7. $a = 8$ cm., $b = 7$ cm., $h_a = 4$ cm.

8. $\angle A = 45°$, $b = 5$ cm., $h_a = 3$ cm. (without protractor).

Problem 8

180. To find the third angle of a triangle if the other two are given.

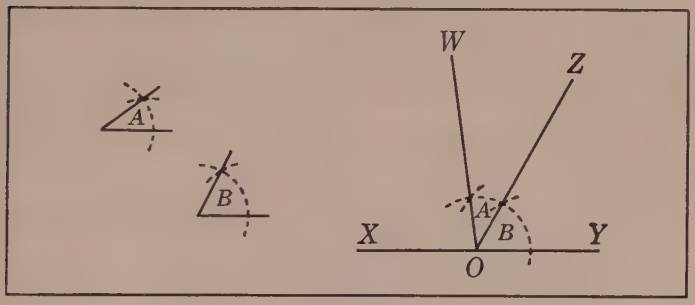


Fig. 252

Given $\angle A$ and $\angle B$, two angles of any $\triangle ABC$.

To find $\angle C$ of $\triangle ABC$.

Directions:

1. Construct the straight angle XOY.
2. On OY with O as vertex construct $\angle YOZ = \angle B$.
3. On OZ with O as vertex construct $\angle ZOW = \angle A$.
4. $\angle WOX$ is the required $\angle C$.

Outline of proof. Give the reasons.

1. $\angle A + \angle B + \angle C = 180°$.
2. $\angle YOZ + \angle ZOW + \angle WOX = 180°$.
3. $\therefore \angle A + \angle B + \angle C = \angle YOZ + \angle ZOW + \angle WOX$.
4. $\therefore \angle C = \angle WOX$.

Exercise. If $\angle A = 37° 18'$ and $\angle B = 52° 47'$, find $\angle C$.

Note. Problem 8 may be solved in two ways. If the number of degrees in each angle is given, the third may be found by arithmetic. If the number of degrees is not given, the third may be found by a geometric construction. It is quite customary in engineering to solve problems by geometric construction.

B Exercises

After solving each of the following exercises give a full discussion (see Exs. 1 and 2, p. 170).

1. Given one acute angle of a right triangle, construct the second.

2. Given one base angle of an isosceles triangle, construct the vertex angle.

3. Given the vertex angle of an isosceles triangle, construct one base angle.

Construct (without protractor) an isosceles triangle having:

4. One leg 5 cm. and one base angle $\frac{3}{8}$ rt. $\angle$.

5. The base 4.3 cm. and the vertex angle 45°.

6. Construct a right triangle, given a and $\angle B$.

7. Construct a right triangle, given $a = 5.4$ cm. and $h_c = 4.2$ cm.

Construct the quadrilateral $ABCD$ (Fig. 253) so that:

8. $AB = 4$ cm., $BC = 7$ cm., $CD = 5$ cm., $AC = 8$ cm., and $\angle C = 60°$.

9. $AB = 6$ cm., $BC = 8$ cm., $CD = 7$ cm., $AD = 9$ cm., and $AC = 10$ cm.

10. $AB = 4.2$ cm., $BC = 6.5$ cm., $CD = 3.7$ cm., $AC = 5.2$ cm., and $BD = 7.8$ cm.

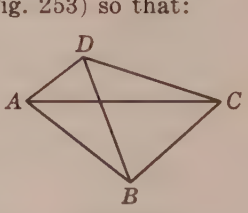


FIG. 253

11. Construct a $\square$, given the sides and one diagonal.

A Exercises

In many of the exercises below, it will be necessary to review the properties of the figures referred to.

Construct an isosceles triangle having:

1. The altitude 4 cm. and one base angle 60°.

2. The altitude 3.7 cm. and the vertex angle $\frac{5}{8}$ rt. $\angle$.

3. The base 4.2 cm. and the perpendicular from one end of the base to the opposite leg 3 cm.

Construct $\triangle ABC$, given:

4. $\angle A = 30°$, $\angle B = 45°$, $h_c = 6.4$ cm.

5. $a = 5.9$ cm., $h_c = 3.2$ cm., $t_c = 4.5$ cm.

6. $h_a = 6$ cm., $\angle A = 60°$, $\angle B = 45°$.

7. a, h_b, m_b. 8. $\angle B$, h_a, t_a. 9. t_c, $\angle C$, $\angle A$.

10. Construct a rectangle, given one diagonal and the angle between the diagonals.

11. Construct a square, given one diagonal.

Construct an isosceles trapezoid, given:

12. The two bases and the altitude.

13. Two consecutive sides and the included angle.

14. The bases and one base angle.

Construct a quadrilateral $ABCD$ with:

15. $AB = 6$ cm., $BC = 3.2$ cm., $CD = 4$ cm., $AD = 2.8$ cm.

16. $AB = 5.7$ cm., $CD = 3.8$ cm., $\angle A = 45°$, $\angle B = 30°$.

Construct a rhombus, given:

17. One side and one diagonal.

18. One side and the altitude.

19. Both diagonals.

20. A diagonal and the angle opposite it.

21. A diagonal and the angle through which it passes.

22. Construct an equilateral triangle, given the altitude.

23. Construct a triangle, given the mid-points of the sides.

24. Construct $\triangle ABC$ with $\angle A = \frac{3}{4}$ rt. $\angle$, $h_b = 5$ cm., $t_a = 4$ cm.

25. Construct $\triangle ABC$ so that $AB = 8.6$ cm., $BC = 6.5$ cm., and the median to AB makes with AB an angle of 45°.

26. Construct an isosceles triangle so that one leg shall be 9.4 cm. and the median to that leg shall make with that leg an angle of $22\frac{1}{2}°$.

27. Construct rt. $\triangle ABC$, given $h_c = 4.6$ cm., $\angle A = \frac{3}{8}$ rt. $\angle$.

PROBLEM 9

181. To construct a triangle, given two sides and the angle opposite one of these sides.

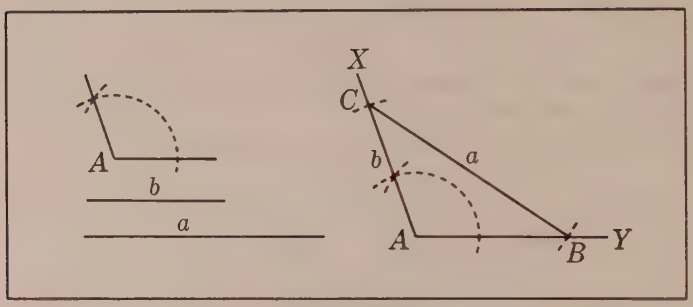


FIG. 254

Given $\angle A$ and segments a and b.

To construct a triangle with $\angle A$ as one angle and segments a and b as sides. Segment a is to be opposite $\angle A$.

Directions:

1. Construct $\angle XAY$ equal to the given $\angle A$.
2. With b as a radius and A as a center cut AX at C.
3. With a as a radius and C as a center cut AY at B.
4. Draw CB.
5. $\triangle ABC$ is the required triangle.

Discussion: The results obtained by the directions for Prob. 9 as given above change as the data given change. Draw the figure for each case indicated below and describe the resulting triangle. Give the reason for the result obtained.

A. When $\angle A$ is obtuse and (1) $a > b$; (2) $a = b$; (3) $a < b$.

B. When $\angle A$ is right and (1) $a > b$; (2) $a = b$; (3) $a < b$.

C. When $\angle A$ is acute (Fig. 255) and

(1) $a > b$;

(2) $a = b$;

(3) $a < b$ and $a > \perp CO$ from C to AY;

(4) $a < b$ and $a = \perp CO$;

(5) $a < b$ and $a < \perp CO$.

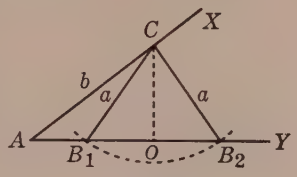


FIG. 255

NOTE. Problem 9, p. 174, is used in trigonometry and surveying.

A Exercises

1. Construct $\triangle ABC$ with $\angle B = 30°$, $a = 10$ cm., $m_a = 4$ cm. Show that more than one triangle is possible and draw each triangle.

2. In making up a problem like Ex. 1, are there any limits to the possible lengths of m_a that must be taken into consideration? If so, what?

3. If, in Ex. 1, $\angle B$ were an obtuse angle, what would be the limits to the lengths of m_a?

4. Construct $\triangle ABC$, given $\angle A$, b, t_c. Give full discussion.

5. Construct $\triangle ABC$, given $\angle A$, b, and m_c. Give full discussion.

6. Construct the fan truss shown in Fig. 256. ABC is an isosceles triangle.

$AD = DC = CE = EB$.

$AF = FD = DG = GC$.

$BK = KE = EH = HC$.

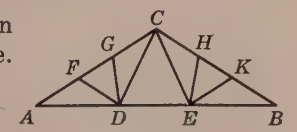


FIG. 256

Suggestion. Construct the isosceles triangle ABC first. To construct $AD = DC$, construct $\angle ACD = \angle CAD$.

7. Construct an isosceles triangle, given one leg and one base angle.

8. Construct a triangle, given two angles and the side opposite one of the given angles.

UNIT III

LOCI

182. Paths of moving points. Life in general presents many illustrations of points moving in a definite path. As a door swings, a point on the knob moves in a definite path; so does a point on the hub or on the tire of a moving wheel. How would you describe the path in such cases? How would you describe the path of a point on the side of a spinning top? Can you think of any other illustrations? The paths in many such cases are more complicated than those you are to study in this unit, but they are very interesting.

From the foregoing it is clear that a point may move so as to fulfill a certain requirement. What this path is depends upon the requirement, such as maintaining a given distance from a given point or as remaining equally distant from two given points.

As far as possible such paths should be located definitely.

How are straight lines located definitely (see §95, p. 81)?

To locate a circle definitely, name its center and its radius.

Ex. 1. Find a point that is one inch† from a fixed point O. Locate ten or twenty such points. If you consider a point as so moving that it will always remain one inch from point O, what will be its path?

Ex. 2. Take A and B any two fixed points. Find ten or twenty points that are as far from A as from B. If you consider a point as so moving that it is always as far from A as from B, what will be its path? Locate this path definitely.

†On the blackboard use any convenient distance.

183. To find the path of a moving point. In some cases the line or set of lines that make up the path or paths can be inferred directly from a knowledge of geometry. In other cases the path must be found by locating a great many positions of the moving point. This is illustrated in the exercises in §182. If positions enough are located, the path may often be inferred from them. The correctness of the inference is then determined by careful proof. This is a method of discovery that is common to all scientific inquiry.

Exercise. Illustrate the method explained above by finding the path of a point that moves so that it is (*a*) always equally distant from two given parallel lines and (*b*) always at a given distance from a given line. (See §37, p. 22.)

B Exercises

NOTE TO THE TEACHER. One or more of the following exercises should be introduced here. The paths of the moving points should be found by experiment. Careful drawings should be required, but no proofs.

1. A ladder stands upright against a wall. Find the path of a point on the middle rung if the foot of the ladder is pulled out until the ladder is flat on the ground.

2. In Fig. 257, *ABCD* is a parallelogram and *l* any line not parallel to either diagonal. Imagine line *l* to move across the parallelogram but always parallel to its original position. Find the path of the mid-point of a segment cut from line *l* by the sides of the parallelogram. Line *l* is a line, not a segment, and may be extended.

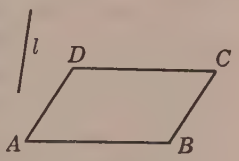


FIG. 257

3. Find the answer to Ex. 2 if *l* is parallel to one of the diagonals.

4. In Fig. 258, the segment AB moves so that its extremities are always on the circle, but its length does not change. Find the path of its mid-point.

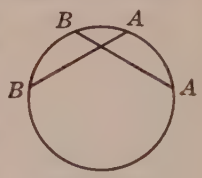


FIG. 258

A segment that has its extremities on the circle is called a **chord** of the circle.

5. Point P (Fig. 259) is any point on a circle. Ray a moves about point P as origin. Find the path of the mid-point of the chord cut from ray a by the circle.

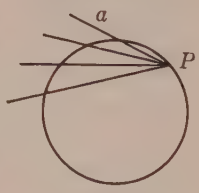


FIG. 259

6. Point P (Fig. 260) is any point within a circle. Ray a moves about point P as origin. Find the path of the mid-point of the segment between P and the circle.

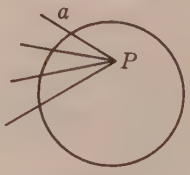


FIG. 260

184. Loci. The path of a point which moves so as to fulfill some given requirement is called a **locus.**

Exercise. Read the exercises in §182, p. 176, and in B Exercises, p. 177, substituting the word "locus" for "path."

185. Locus theorems. A theorem stating that a line or set of lines is a locus must state two facts. It

(1) must state clearly the given requirement that the given point must obey;

(2) must locate definitely the path of the moving point.

Ex. 1. In each of the theorems on p. 179 find: (*a*) the law that the moving point must obey and (*b*) the path of the moving point.

Ex. 2. State each theorem given on p. 179 so that the statement of what the path is comes first, thus: The circle with the given point as center . . . . is the locus of . . . . .

1. The locus of points at a given distance from a given point is a circle with the given point as center and the given distance as radius.

2. The locus of points equally distant from two given points is the perpendicular bisector of the segment joining the given points (see §27, p. 18).

3. The locus of points equally distant from two given parallel lines is a line parallel to the given lines and passing through the mid-point of the perpendicular between them (see §37, p. 22).

4. The locus of points at a given distance from a given straight line consists of two lines each parallel to the given line at the given distance from it (see §37, p. 22).

5. The locus of points equally distant from the sides of an angle is the bisector of the angle (see §37, p. 22).

186. The formal definition of a locus. A line or group of lines is called a locus if they contain all points that fulfill some given requirement and contain no other points.

187. Proofs of locus theorems. After the locus has been found by some method (see §183, p. 177), it is necessary to give a complete formal proof that the line or set of lines really constitutes the locus.

In order to prove that a line or set of lines is the locus of a point which moves so as to fulfill some given requirement, prove that:

a. If a point is on the line or set of lines, it fulfills the requirement, and

b. If a point fulfills the requirement, it is on the line or set of lines.

NOTE. It is immaterial in which order *a* and *b* are proved.

188. Formal proofs will be given for theorems 2 and 5 quoted in §185. The other three may be assumed or proved as your teacher desires.

THEOREM 35*

189. The perpendicular bisector of a segment is the locus of points equally distant from the ends of the segment.

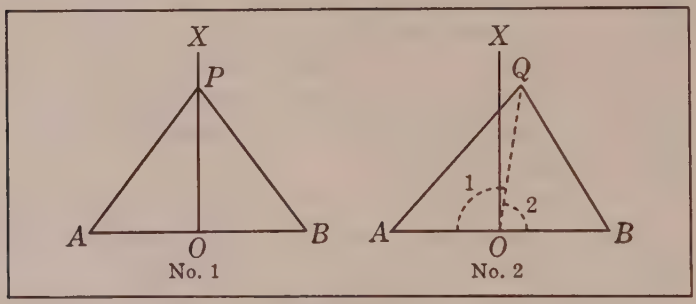


FIG. 261

Hypothesis: XO is the perpendicular bisector of AB.

Conclusion: XO is the locus of points equally distant from A and B.

Analysis: To prove XO the locus of points equally distant from A and B, prove that:

I. If a point is in XO, it is equally distant from A and B.

II. If a point is equally distant from A and B, it is in XO.

For I (Fig. 261, No. 1):

Hypothesis: XO is the perpendicular bisector of AB, point P is in XO.

Conclusion: P is equally distant from A and B.

Give the analysis and proof.

For II (Fig. 261, No. 2):

Hypothesis: XO is the perpendicular bisector of AB, Q is equally distant from A and B.

Conclusion: Q is in XO.

Analysis and construction:

1. To prove that Q lies in XO, join Q and O, the midpoint of AB, and prove that QO and XO coincide.
2. To prove that QO and XO coincide, show that QO and XO are both $\perp AB$ at O.
3. To prove $QO \perp AB$, prove $\angle 1 = \angle 2$.
4. " " $\angle 1 = \angle 2$, { draw QA and QB and prove $\triangle AOQ \cong \triangle BOQ$.

Outline of proof. Give all reasons.

1. *a.* $\triangle AOQ \cong \triangle BOQ$.
   b. $\therefore \angle 1 = \angle 2$.
2. $\therefore QO \perp AB$ at O.
3. $XO \perp AB$ at O.
4. $\therefore XO$ and QO coincide.
5. $\therefore Q$ lies in XO.

Conclusion: XO is the locus of points equally distant from A and B because:

I. If a point is in XO, it is equally distant from A and B, and

II. If a point is equally distant from A and B, it lies in XO.

190. The two facts used in proving Th. 35, may be stated generally as follows:

I. If a point is in the perpendicular bisector of a segment, it is equally distant from the ends of the segment.

II. If a point is equally distant from the ends of a segment, it is in the perpendicular bisector of the segment.

191. COR. **If two points are each equally distant from the extremities of a segment, the line passing through these points is the perpendicular bisector of the segment.**

THEOREM 36*

192. The bisector of an angle is the locus of points equally distant from the sides of the angle.

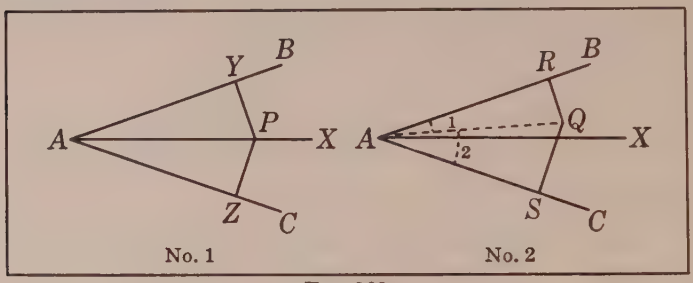


FIG. 262

Hypothesis: AX is the bisector of $\angle A$.

Conclusion: AX is the locus of points equally distant from AB and AC.

Analysis: To prove AX the locus of points equally distant from AB and AC, prove that:

I. If a point is in AX, it is equally distant from AB and AC.

II. If a point is equally distant from AB and AC, it lies in AX.

For I (Fig. 262, No. 1):

Hypothesis: AX is the bisector of $\angle A$. P is in AX.

Conclusion: P is equally distant from AB and AC.

Analysis: To prove P equally distant from AB and AC, draw PY from $P \perp AB$ and PZ from $P \perp AC$ and prove $PY = PZ$.

For II (Fig. 262, No. 2):

Hypothesis: Q is equally distant from AB and AC. AX bisects $\angle A$.

Conclusion: Q lies in AX.

Analysis and construction:

1. To prove that Q lies in AX, join Q and A and prove that AQ and AX coincide.
2. To prove that AQ and AX coincide, show that both AQ and AX bisect $\angle A$.
3. To prove that AQ bisects $\angle A$, prove $\angle 1 = \angle 2$.
4. " " $\angle 1 = \angle 2$, draw QR from $Q \perp AB$ and QS from $Q \perp AC$ and prove $\triangle QAR \cong \triangle QAS$.

Outline of proof. Give all reasons.

1. *a.* $\triangle QAR \cong \triangle QAS$.
   b. $\therefore \angle 1 = \angle 2$.
2. $\therefore AQ$ bisects $\angle A$.
3. AX bisects $\angle A$.
4. $\therefore AQ$ and AX coincide.
5. $\therefore Q$ lies in AX.

Conclusion: AX is the locus of points equally distant from AB and AC because:

I. If a point is in AX, it is equally distant from AB and AC.

II. If a point is equally distant from AB and AC, it lies in AX.

193. The two facts used in proving Th. 36 may be stated generally as follows:

I. If a point is in the bisector of an angle, it is equally distant from the sides of the angle.

II. If a point is equally distant from the sides of an angle, it lies in the bisector of the angle.

194. COR. **The locus of points equally distant from two intersecting straight lines consists of the bisectors of the angles formed by these lines.**

195. Memorize the five elementary locus theorems given on p. 179. In each case:

a. Draw the figure.

b. State the law that the moving point must obey.

c. State the path along which the point moves.

d. State the hypothesis and conclusion.

e. State the two facts necessary for a complete proof in each case. State these facts (1) in general and (2) in terms of the letters of your figure.

C Exercises

In the following exercises it is required to find a point which must fulfill a certain condition. This condition calls for the construction of a locus. Determine in each case what this condition is and what locus must be drawn. Draw the figure and give all details as for Ex. 1.

1. Find in a given line a point that is equally distant from two given points.

Let l represent the given line and A and B the given points (Fig. 263).

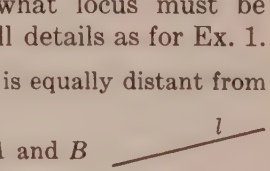


FIG. 263

Analysis and directions: Call the required point X.

I. Since X must be equally distant from A and B, it must lie in the perpendicular bisector of AB (Th. 35).

II. Since X must also lie in line l, it must be the intersection of l and the perpendicular bisector of AB.

Complete the figure and find X.

Proof: X is equally distant from A and B. Why?

Discussion: How many points can be found as required? Are there any special positions of the given line or the given points that will alter the results? State in full and give reasons.

Find in a given line a point that is:

2. Equally distant from two given parallels.

3. At a given distance from a second given line.

4. Equally distant from two given intersecting lines.

B Exercises

The following exercises are solved by the intersection of two loci.

1. Find a point that shall be equally distant from two given parallel lines and at the same time equally distant from two given points.

Let h and k represent the two given parallels and A and B the two given points (Fig. 264).

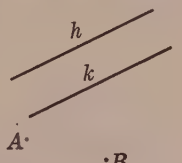


FIG. 264

Analysis and directions: Call the required point X.

I. Since X must be equally distant from h and k, it must lie on a line that is parallel to h and k and bisecting a perpendicular between them.

II. Since X must be equally distant from A and B, it must lie on the perpendicular bisector of AB.

Show how to find the required point, prove that it fulfills both requirements, and give the discussion.

2. Find a point that shall be at a given distance from a given line and at the same time equally distant from two given points.

3. Find a point that shall be equally distant from two given parallel lines and at the same time at a given distance from a given line.

4. Find a point that shall be equally distant from two given points and at the same time equally distant from two given intersecting lines.

THEOREM 37 (*Optional*)

196. The perpendicular bisectors of the sides of a triangle meet in a point that is equally distant from the vertices of the triangle.

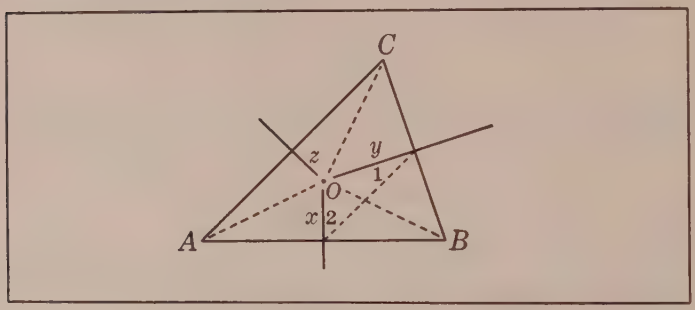


FIG. 265

Hypothesis: $\triangle ABC$ is any triangle. Lines x, y, and z are the $\perp$ bisectors of AB, BC, and CA respectively.

Conclusion: x, y, and z meet at a point which is equally distant from A, B, and C.

Analysis: To prove x, y, and z meet at a point, prove

A. That x and y intersect at some point (as at O).

B. That the intersection of x and y is in z.

Outline of proof. Give the reasons.

A. 1. If x and y are $\|$, $\angle 1 + \angle 2 = 2$ rt. $\angle$s.
2. But $\angle 1 < 1$ rt. $\angle$ and $\angle 2 < 1$ rt. $\angle$.
3. $\therefore \angle 1 + \angle 2 < 2$ rt. $\angle$.
4. $\therefore x$ and y are not $\|$ and intersect.

B. 1. *a*. O lies in line x.
b. $\therefore OA = OB$.
2. Similarly $OB = OC$.
3. $\therefore OA = OC$.
4. $\therefore O$ is in line z.

THEOREM 38 (*Optional*)

197. The bisectors of the angles of a triangle meet at a point that is equally distant from the sides of the triangle.

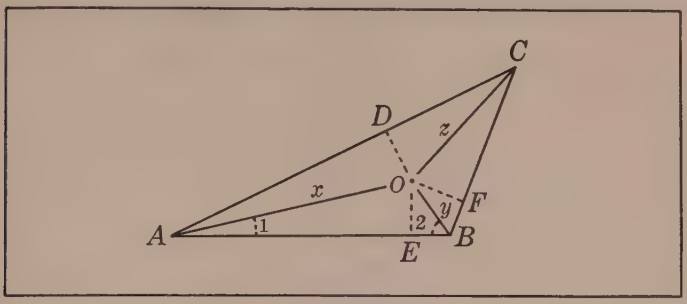


FIG. 266

Hypothesis: $\triangle ABC$ is any triangle. x, y, and z are the bisectors of $\measuredangle A$, B, and C respectively.

Conclusion: x, y, and z meet in a point equally distant from the sides.

Analysis: To prove x, y, and z meet at a point, prove

A. That x and y intersect at some point (as at O).

B. That the intersection of x and y is on z.

Outline of proof. Give the reasons.

A. 1. If $x \parallel y$, then $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$.

2. But $\angle 1 < \angle A$ and $\angle 2 < \angle B$.

3. $\therefore \angle 1 + \angle 2 < 2$ rt. $\measuredangle$.

4. $\therefore$ x and y are not $\parallel$ and intersect.

B. 1. *a*. O lies in line x.

b. $\therefore$ $\perp$ from O to AC equals $\perp$ from O to AB ($OD = OE$).

2. Similarly $\perp$ from O to AB equals $\perp$ from O to BC ($OE = OF$).

3. $\therefore OD = OF$.

4. $\therefore$ O lies on z.

THEOREM 39 (*Optional*)

198. The lines through the vertices of a triangle perpendicular to the opposite sides meet in a point.

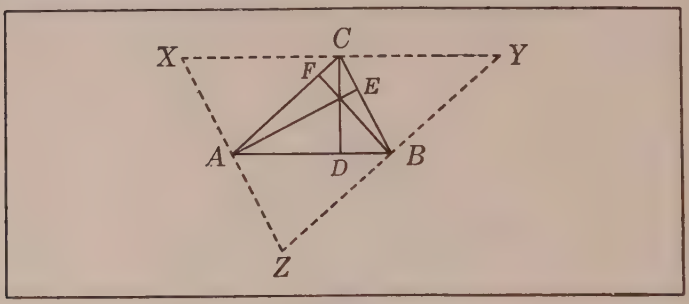


FIG. 267

Hypothesis: ABC is any $\triangle$; $CD \perp$ from C to AB; $BF \perp$ from B to CA; $AE \perp$ from A to BC.

Conclusion: CD, BF, and AE meet in a point.

Analysis: To prove that CD, BF, and AE meet in a point, draw XY through $C \parallel AB$, YZ through $B \parallel AC$, and ZX through $A \parallel BC$ and prove CD, BF, and AE the perpendicular bisectors of the sides of $\triangle XYZ$.

Outline of proof. Give the reasons.

1. $CD \perp XY$.

2 *a*. $ABCX$ is a $\square$.

b. $\therefore AB = XC$.

c. Similarly $AB = CY$.

d. $\therefore CX = CY$ and C is the mid-point of XY.

e. $\therefore CD$ is the $\perp$ bisector of XY.

3. Similarly FB is the $\perp$ bisector of YZ.
and AE is the $\perp$ bisector of ZX.

4. $\therefore CD$, BF, and AE meet in a point.

Survey Test

1. When does a point move in a definite path?

2. (*a*) How are straight lines located definitely? (*b*) How are circles located definitely?

3. State two ways in which a locus of a moving point may be found.

4. Define a locus in two ways.

5. (*a*) What two facts must be stated in a locus theorem? (*b*) What two facts must be proved in a locus theorem?

6. What is the locus (*a*) of points at a given distance from a given point? (*b*) of points equally distant from two given points? (*c*) of points equally distant from two given parallel lines? (*d*) of points at a given distance from a given straight line? (*e*) of points equally distant from the sides of an angle?

7. (*a*) The perpendicular bisector of a segment is the locus of what points? (*b*) State the two facts used in proving this theorem.

8. State the corollary of the theorem quoted in answer to 7*a*.

9. (*a*) The bisector of an angle is the locus of what points? (*b*) State the two facts used in proving this theorem.

10. State the corollary of the theorem quoted in answer to 9*a*.

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*
1	176	6	179
2	81	7	180, 181
3	177	8	181
4	178, 179	9	182, 183
5	178, 179	10	183

William Thompson

FIG. 268. **Old Roman aqueduct near Nîmes, France.** Notice the use of semicircles.

UNIT IV

CIRCLES

199. Circles in the world about you. Circles are extensively used in ornament and industry. They are everywhere about you.

Ex. 1. Turn to p. 214 and notice the tile design in Fig. 297. Make at least one of the drawings suggested in the note and bring it to class. Can you find in this book any other tile or mosaic floor design showing circles? Give the page and number of the figure.

Ex. 2. Bring into class a picture of a tile decorated in this way or a drawing of one that you have seen in some public building. Tell exactly where you found it.

The designs found in ornamental ironwork are often based on circles. Many of them are very old. You will find them on grilles used for registers, steam radiator covers, and screening purposes generally. Look at the screens inclosing the tellers' cages in banks, at elevator gates, and at railings for stairs and balconies. On page 402, Fig. 567 is a design from an old Roman floor that is frequently used in grilles. (See also Fig. 350, page 240.)

Ex. 3. Bring into class some picture that you have found showing circles in ironwork or a drawing of such a design that you have seen. Tell exactly where you found it.

A large number of church window designs, especially those of the European cathedrals and the older churches in this country, are based on circles. For examples, see page 219. The trefoil (p. 401) and the quadrifoil had symbolic religious meanings in very early times.

Miscellaneous examples of the uses of circles are varied and numerous. You will often find them in all-over lace patterns, while the many pointed stars found on cut

glass are derived from circles (see pp. 252 and 388). The first arch used was by the old Romans and was semicircular (p. 190). Cross sections of moldings used by carpenters and builders are often made of parts of circles (p. 218). Machinery generally is full of circles in the form of wheels. On page 405 is a cross section of a sewer made of parts of circles. Do you know of other examples?

200. Review the definitions of **circle, radius, diameter,** and **arc** (§66, p. 45) and illustrate by a figure. A circle whose center is O is written $\odot O$ and is read circle O.

Post. 25. The diameter of a circle is twice its radius.

To draw a circle, it is necessary to know two things, its center and its radius.

Post. 26. A circle is located definitely if its center and its radius are known.

201. Two circles or two arcs that can be made to coincide are said to be **congruent.**

Post. 27. Circles with equal radii are congruent.

Post. 28. Congruent circles have equal radii.

Exercise. Draw two circles with equal radii. Cut them out and place together with the same pin through their centers. Can arcs of circles with different radii be congruent?

202. A segment joining two points on a circle is called a **chord** of the circle.

A chord joining the ends of an arc is called the **chord of the arc.** Every arc has one and only one chord. Every chord, however, has two arcs. If the chord is a diameter, the two arcs are congruent and are called **semicircles.**

Post. 29. Every diameter bisects the circle.

Exercise. Illustrate Post. 29 by drawing a circle and any diameter. Fold the circle on this diameter; hold to the light.

203. If a chord of a circle is not a diameter, its two arcs are unequal. The larger arc of a chord is called the **major arc** and the smaller one the **minor arc.** Illustrate by a figure.

Post. 30. Between the same two points on a circle there is one and only one minor arc, provided these two points are not the ends of a diameter.

204. An angle with its vertex at the center of a circle is called a **central angle.** The sides of the angle cut off two arcs on a circle. The minor arc cut off by the sides of the angle is said to be its **intercepted arc.**

Post. 31. In the same circle or in congruent circles, equal central angles intercept equal minor arcs.

Post. 32. In the same circle or in congruent circles, equal minor arcs intercept equal central angles.

Illustrate Posts. 31 and 32 by drawing figures on thin paper. Place the centers together and hold to the light.

Exercise. Turn to circle figures elsewhere in the book and find illustrations of the definitions given in §§201–204.

205. Draw a figure to illustrate each of the postulates that follow. State the hypothesis and conclusion for each.

Post. 33. If a line passes through a point within a circle, the line and the circle intersect in two and only two points.

Post. 34. A segment joining a point without a circle and the center is longer than the radius.

Post. 35. If a segment that has one end at the center of a circle is longer than the radius, it extends without the circle and cuts the circle but once.

Post. 36. A segment joining a point within a circle and the center is shorter than the radius.

Post. 37. If a segment that has one end at the center of a circle is shorter than the radius, it lies wholly within the circle.

TOPIC 1. CIRCLES AND RELATED LINES

THEOREM 40

206. In the same circle or in congruent circles,

A. Equal chords intercept equal central angles.

B. Equal central angles intercept equal chords.

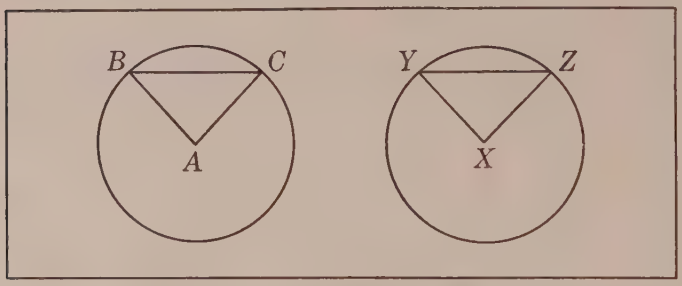


FIG. 269

A. Hypothesis: $\odot A \cong \odot X$. Chord BC = chord YZ.
Conclusion: $\angle A = \angle X$.
Analysis:
To prove $\angle A = \angle X$, prove $\triangle ABC \cong \triangle XYZ$.

B. Hypothesis: $\odot A \cong \odot X$. $\angle A = \angle X$.
Conclusion: Chord BC = chord YZ.
Analysis:
To prove $\overline{BC} = \overline{YZ}$, prove $\triangle ABC \cong \triangle XYZ$.

207. Intersecting circles. The segment joining the points of intersection of two circles is called the **common chord** of the two circles. The line passing through the centers of two circles is called the **line of centers.**

Ex. 1. If any two circles intersect, the line of centers bisects the common chord at right angles (Th. 5, p. 73).

Ex. 2. If two congruent circles intersect, the line of centers and the common chord bisect each other at right angles.

Theorem 41

208. In the same circle or in congruent circles,

A. Equal chords have equal minor arcs.

B. Equal minor arcs have equal chords.

A. Analysis: To prove $\overset{\frown}{BC} = \overset{\frown}{YZ}$, prove $\angle A = \angle X$ (by Th. 40A). Then the arcs are equal by Post. 31, p. 193.

B. Analysis: To prove $\overline{BC} = \overline{YZ}$, prove $\angle A = \angle X$ (by Post 32). Then the chords are equal by Th. 40B.

C Exercises

1. If, in Fig. 270, $\overline{AD} = \overline{BC}$, prove $\overset{\frown}{AC} = \overset{\frown}{BD}$ and $\overset{\frown}{DAB} = \overset{\frown}{CBA}$.

2. If, in Fig. 270, $\overset{\frown}{AC} = \overset{\frown}{BD}$, prove $\overline{AD} = \overline{BC}$.

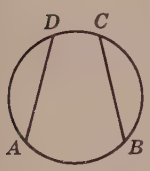


FIG. 270

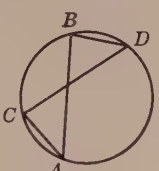


FIG. 271

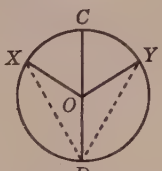


FIG. 272

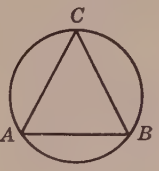


FIG. 273

3. If, in Fig. 271, $\overline{AB} = \overline{CD}$, prove $\overline{AC} = \overline{BD}$.

4. If, in Fig. 271, $\overline{AC} = \overline{BD}$, prove $\overline{AB} = \overline{CD}$.

5. If, in Fig. 272, O is the center of the circle and CO bisects $\angle XOY$, prove $\overset{\frown}{XD} = \overset{\frown}{YD}$.

6. If, in Fig. 272, O is the center of the circle and $\overline{DX} = \overline{DY}$, prove that CO bisects $\angle XOY$.

7. In Fig. 273, ABC is any triangle with its vertices on the circle. If $\angle A = \angle B$, prove $\overset{\frown}{AC} = \overset{\frown}{BC}$.

8. If, in Fig. 274, AC is a diameter of the circle and $\overset{\frown}{AB} = \overset{\frown}{CD}$, prove $ABCD$ a $\square$.

FIG. 274

NOTE. It will be proved later that $ABCD$ is a rectangle.

THEOREM 42*

209. A line through the center of a circle perpendicular to a chord bisects the chord and its arcs.

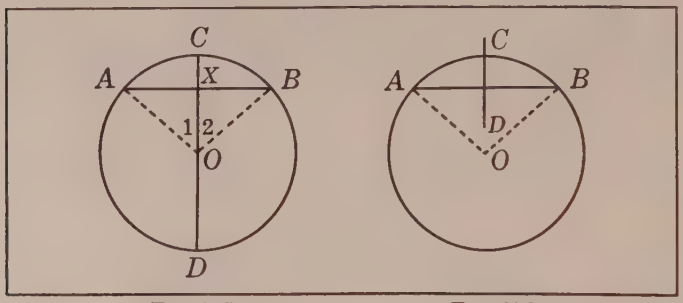


FIG. 275 FIG. 276

Hypothesis: $\odot O$ is any circle. $\overline{AB}$ is any chord of $\odot O$. CD is $\perp AB$ and passes through the center (Fig. 275).

Conclusion: CD bisects the $\overline{AB}$, the $\overset{\frown}{AB}$, and the $\overset{\frown}{ADB}$.

Analysis:

1. To prove CO bisects $\overline{AB}$, prove $AX = XB$.
2. " " $AX = XB$, " $\triangle AXO \cong \triangle BXO$.
3. " " CD bisects $\overset{\frown}{AB}$, " $\overset{\frown}{AC} = \overset{\frown}{CB}$.
4. " " $\overset{\frown}{AC} = \overset{\frown}{CB}$, " $\angle 1 = \angle 2$.
5. " " CD bisects $\overset{\frown}{ADB}$, " $\overset{\frown}{AD} = \overset{\frown}{DB}$.
6. " " $\overset{\frown}{AD} = \overset{\frown}{DB}$, " $\angle AOD = \angle BOD$.

210. COR. The perpendicular bisector of a chord passes through the center of the circle.

Hypothesis: CD is the $\perp$ bisector of $\overline{AB}$ (Fig. 276).

Conclusion: CD passes through O.

Analysis:

1. To prove that CD passes through O, prove that O lies on CD by II, §190, p. 181.
2. $\therefore$ show that $AO = BO$.

PROBLEM 10

211. Construct a circle that shall pass through three given points not in the same straight line.

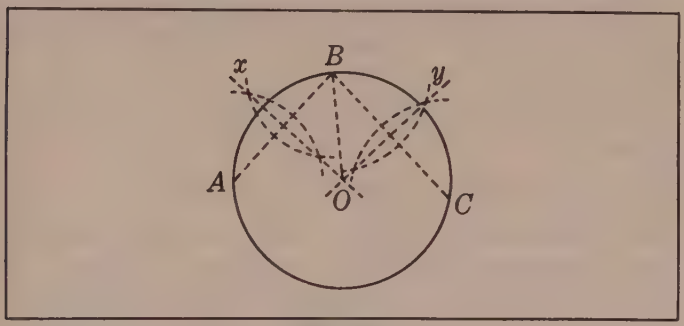


FIG. 277

Given the three points A, B, and C not in the same straight line.

To construct a circle through points A, B, and C.

Analysis: AB and BC will be chords of the circle. Use §210.

Directions:

1. Draw AB and BC.
2. Construct x, the $\perp$ bisector of AB, and y, the $\perp$ bisector of BC.
3. Let x and y intersect at O.
4. With OB as radius and O as center draw the circle.

Proof: Apply §210.

212. *Post. 38.**† *Through three given points not in the same straight line one circle and only one can be drawn.*

†*To the teacher:* Post. 38 is really a corollary of Th. 37, p. 186. But, as Th. 37 is optional, Post. 38 may be assumed here if Th. 37 was omitted; otherwise it should be proved.

213. Definite location of circles. You now have two methods of locating circles definitely.

A circle is located definitely if you know

(1) Its center and radius (Post. 26, p. 192).

(2) Three points through which it passes (Post. 38, p. 197).

Post. 39. Two circles cannot intersect in more than two points.

Post. 39 is a direct consequence of Post. 38.

214. Circumscribed circles and inscribed polygons. If a circle passes through the vertices of a polygon, the circle is said to be **circumscribed about the polygon.** If the vertices of a polygon lie on a circle, the polygon is said to be **inscribed in the circle.** Evidently, then, a circle is circumscribed about any polygon if that polygon is inscribed in it.

Exercise. *Show how to circumscribe a circle about a triangle.*

C Exercises

1. In any circle, draw a chord AB and a diameter perpendicular to it meeting the circle at C and D. Prove $\triangle CAB$ and $\triangle DAB$ isosceles.

2. Fig. 278 shows two circles with the same center. Segment AB cuts both circles. Prove $AX = BY$.

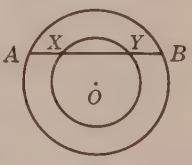


FIG. 278

Suggestion. Draw a line from O perpendicular to AB and subtract equal segments from equal segments. Do not use triangles.

3. A segment from the center of a circle to the mid-point of an arc is the perpendicular bisector of the chord of the arc.

4. A line from the center of a circle through the mid-point of a chord is perpendicular to the chord and bisects the arc of the chord.

THEOREM 43*

215. If, in the same circle or in congruent circles, perpendiculars from the center to two chords are equal, the chords are equal.

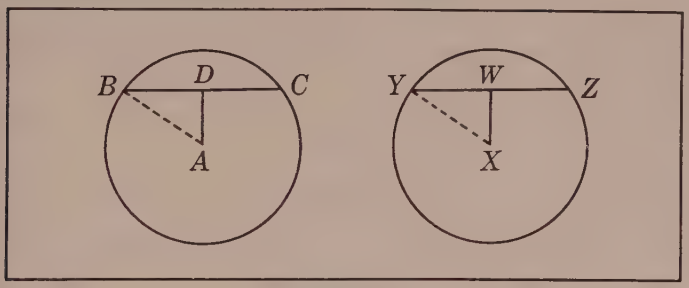


FIG. 279

Hypothesis: $\odot A \cong \odot X$, $AD \perp \overline{BC}$ from A, $XW \perp \overline{YZ}$ from X, and $AD = XW$.

Conclusion: $\overline{BC} = \overline{YZ}$.

Analysis and construction:

1. To prove $\overline{BC} = \overline{YZ}$, prove BD and YW halves of $\overline{BC}$ and $\overline{YZ}$ respectively and prove $BD = YW$ (Th. 42, p. 196, and Ax. 3, p. 30).
2. To prove $BD = YW$, draw AB and XY and prove $\triangle ABD \cong \triangle XYW$ (Th. 23, p. 122).

THEOREM 44*

216. In the same circle or in congruent circles perpendiculars from the center to two equal chords are equal.

Analysis and construction (Fig. 279):

1. To prove $AD = XW$, draw AB and XY and prove $\triangle ABD \cong \triangle XYW$ (Th. 23, p. 122).
2. To prove $\triangle ABD \cong \triangle XYW$, prove $BD = YW$ (Th. 42, p. 196, and Ax. 4, p. 30).

B Exercises

1. Two parallel chords in a circle intercept equal arcs.

Suggestion. Draw a diameter perpendicular to one of the two chords. Prove it perpendicular to the other. Then subtract equal arcs from equal arcs.

2. A diameter bisecting one of two parallel chords bisects the other.

3. The perpendicular bisector of a chord bisects any chord parallel to the given chord.

Suggestion. Let XY be the perpendicular bisector of AB. Prove XY a diameter and perpendicular to the other chord.

4. A segment joining the mid-points of the two arcs of a chord is a diameter of the circle (apply Th. 5, p. 73, and §210, p. 196).

5. Two parallel chords through the ends of a diameter are equal (apply Th. 43, p. 199).

6. If two equal chords intersect within a circle, they make equal angles with the segment joining the point of intersection and the center of the circle.

7. If two lines intersect at a point within a circle and make equal angles with the segment joining the point of intersection and the center of the circle, the chords cut off are equal.

217. (*Optional.*) Theorems concerning inequalities in circles are based on the two following postulates and Topic 7, page 162.

Post. 40. In the same circle or in congruent circles, if two central angles are unequal, the minor arc intercepted by the greater angle is greater than the minor arc intercepted by the lesser angle.

Post. 41. In the same circle or in congruent circles, if two minor arcs are unequal, the angle intercepted by the greater arc is greater than the angle intercepted by the lesser arc.

THEOREM 45 (*Optional*)

218. In the same circle or in congruent circles, if two chords are unequal, the central angle intercepted by the greater chord is greater than the central angle intercepted by the lesser chord.

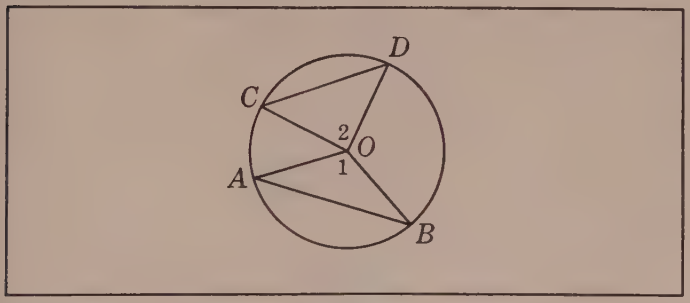


FIG. 280

Hypothesis: $\odot O$ is any circle, $\overline{AB}$ intercepts the central $\angle 1$, and $\overline{CD}$ intercepts the central $\angle 2$. $\overline{AB} > \overline{CD}$.

Conclusion: $\angle 1 > \angle 2$.

Suggestion. Apply Th. 34, p. 167. In $\triangle AOB$ and $\triangle COD$, $AO = CO$, $BO = DO$, and $AB > CD$. No analysis is needed.

THEOREM 46 (*Optional*)

219. In the same circle or in congruent circles, if two central angles are unequal, the chord intercepted by the greater central angle is greater than the chord intercepted by the lesser central angle (Fig. 280).

Hypothesis: $\odot O$ is any circle. Central $\angle 1$ intercepts $\overline{AB}$ and central $\angle 2$ intercepts $\overline{CD}$. $\angle 1 > \angle 2$.

Conclusion: $\overline{AB} > \overline{CD}$.

Suggestion Apply Th. 33, p. 166. In $\triangle AOB$ and $\triangle COD$ $AO = CO$, $BO = DO$, and $\angle 1 > \angle 2$. No analysis is needed.

THEOREM 47 (*Optional*)

220. In the same circle or in congruent circles, if two chords are unequal, the minor arc of the greater chord is greater than the minor arc of the lesser chord.

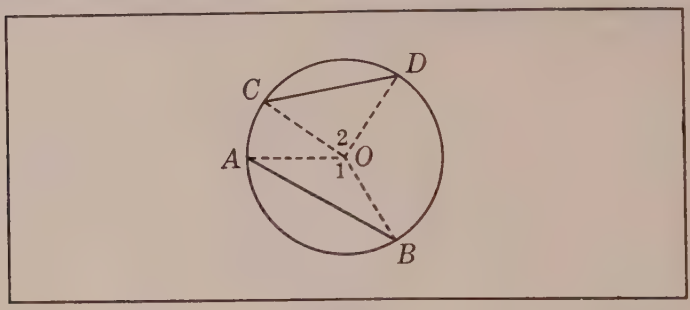


FIG. 281

Hypothesis: $\odot O$ is any circle. $\overline{AB} > \overline{CD}$, $\overset{\frown}{AB}$ is the minor arc of $\overline{AB}$, and $\overset{\frown}{CD}$ is the minor arc of $\overline{CD}$.

Conclusion: $\overset{\frown}{AB} > \overset{\frown}{CD}$.

Analysis: Draw CO, DO, AO, and BO and prove $\angle 1 > \angle 2$ (by Th. 45, p. 201). Then $\overset{\frown}{AB} > \overset{\frown}{CD}$ (by Post. 40, p. 200).

THEOREM 48 (*Optional*)

221. In the same circle or in congruent circles, if two minor arcs are unequal, the chord of the greater arc is greater than the chord of the lesser arc (Fig. 281).

Hypothesis: $\odot O$ is any circle, minor $\overset{\frown}{AB}$ > minor $\overset{\frown}{CD}$, $\overline{AB}$ is the chord of $\overset{\frown}{AB}$, and $\overline{CD}$ is the chord of $\overset{\frown}{CD}$.

Conclusion: $\overline{AB} > \overline{CD}$.

Analysis: Draw CO, DO, AO, and BO and prove $\angle 1 > \angle 2$ by Post. 41, p. 200. Then $\overline{AB} > \overline{CD}$ by Th. 46.

THEOREM 49 (*Optional*)

222. In the same circle or in congruent circles, if two chords are unequal, the greater chord is at the less distance from the center.

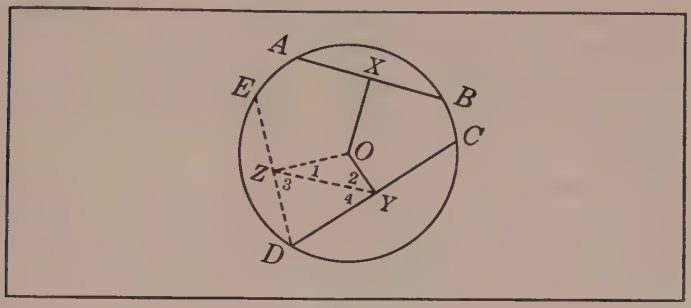


FIG. 282

Hypothesis: $\odot O$ is any circle. OX is $\perp \overline{AB}$ from O and OY is $\perp \overline{CD}$ from O. $\overline{CD} > \overline{AB}$.

Conclusion: $OY < OX$.

Analysis and construction:

1. To prove $OY < OX$, apply Th. 31, p. 164.
2. Construct $DE = AB$ from D. Draw OZ from $O \perp DE$ and prove $OZ = OX$ and $OY < OZ$.
3. To prove $OY < OZ$, draw ZY and prove $\angle 1 < \angle 2$.
4. " " $\angle 1 < \angle 2$, prove $\begin{cases} \angle OZD = \angle OYD \text{ and} \\ \angle 3 > \angle 4. \end{cases}$
5. " " $\angle 3 > \angle 4$, prove $DY > DZ$ and use Th. 32.

THEOREM 50 (*Optional*)

223. In the same circle or in congruent circles, if two chords are unequally distant from the center, the chord that is nearer the center is the greater (Fig. 282).

Analysis and proof are similar to those for Th. 49.

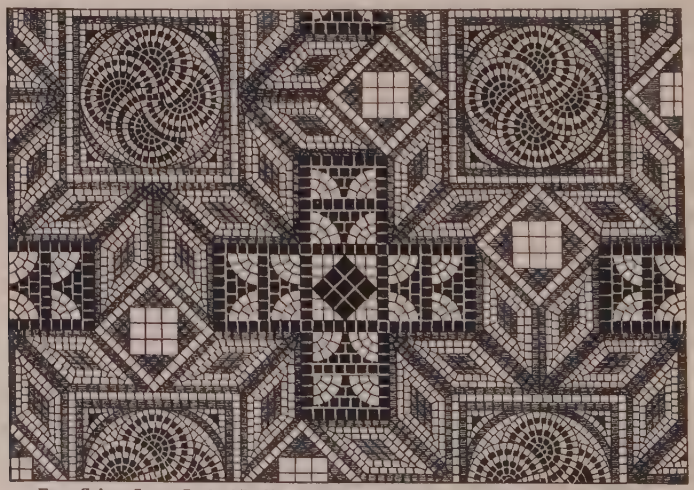
224. Tangents. A line that touches a circle at one point, but does not cut it, is called a **tangent** to the circle.

The point at which the tangent touches the circle is called the **point of contact,** or the **point of tangency.**

225. Inscribed circles and circumscribed polygons. If a circle is tangent to the sides of a polygon, it is said to be **inscribed in the polygon.** If the sides of a polygon are tangent to a circle, the polygon is said to be **circumscribed about the circle.**

226. Two circles are said to be **tangent** if they are tangent to the same line at the same point. They are said to be **tangent internally** if one is within the other. They are said to be **tangent externally** if neither circle is within the other. The point of contact of the common tangent is the **point of contact of the two circles.**

Exercise. Find in Fig. 283 illustrations of tangents to circles and of tangent circles.



From Sykes, *Source Book of Problems for Geometry* Courtesy Allyn & Bacon

FIG. 283. **Mosaic floor design**

THEOREM 51

227. A line which is perpendicular to a radius at its outer extremity is a tangent to the circle.

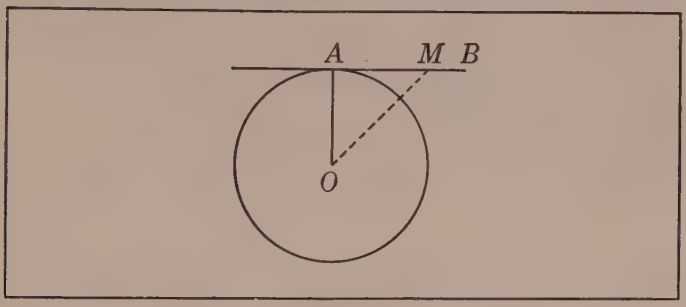


FIG. 284

Hypothesis: $\odot O$ is any circle; OA is any radius; line $AB \perp OA$ at A.

Conclusion: AB is tangent to $\odot O$ at A.

Analysis and construction:

1. To prove that AB is tangent to $\odot O$ at A, prove that all points in AB except A lie outside the circle.
2. To prove that any point in AB other than A, such as M, lies outside the circle, join O and M and prove $OM > OA$.

Proof:

STATEMENTS	REASONS
1. $OM > OA$.	1. Post. 12, p. 31.
2. $\therefore M$ lies outside $\odot O$.	2. Post. 35, p. 193.
3. All points except A lie outside $\odot O$.	3. Since M is any point in AB other than A.
4. $\therefore AB$ is tangent to $\odot O$.	4. Why?

Exercise. Show how to construct a tangent to a circle from a point on the circle.

THEOREM 52

228. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

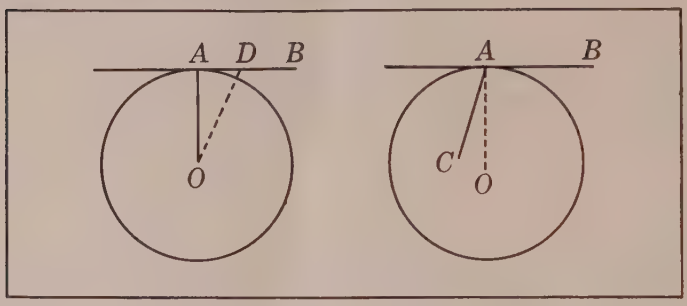


FIG. 285 FIG. 286

Hypothesis: $\odot O$ is any circle, AB is tangent to $\odot O$ at A. OA is a radius drawn to the point of contact A.

Conclusion: $OA \perp AB$ (Fig. 285).

Analysis and construction:

1. Either $OA \perp AB$ or OA is not $\perp AB$. Suppose OA is not $\perp AB$.
2. A $\perp$ from O to AB is possible.
3. $\therefore$ join O to D, any point in AB, and show that OD cannot be $\perp AB$.

Outline of proof. Give the reasons.

1. Suppose $OD \perp AB$.
2. Then OD would be less than the radius OA.
3. Then OD would lie wholly within $\odot O$ (Post. 37, p. 193).
4. But D lies without $\odot O$.
5. $\therefore$ OD is not $\perp AB$.
6. No segment from O to AB can be $\perp AB$ except OA.
7. $\therefore$ OA is $\perp AB$.

229. Cor. (*Optional*).† **A perpendicular to a tangent at the point of contact passes through the center of the circle** (Fig. 286).

Analysis and construction:

1. To prove that AC passes through O, prove that AC coincides with a line that does pass through O.
2. $\therefore$ draw AO and prove that AO and AC coincide.
3. To prove that AO and AC coincide, show that they are both $\perp AB$ at A.

Proof:

STATEMENTS	REASONS
1. $AC \perp AB$ at A.	1. Given.
2. $AO \perp AB$ at A.	2. Th. 52.
3. AC and AO coincide.	3. Post. 10, p. 31.
4. AC passes through O.	4. Why?

C Exercises

1. Two tangents at the ends of a diameter are parallel.

2. AC is a diameter of a circle. AB is tangent to the circle at A. CD is drawn through $C \parallel AB$. Prove CD tangent to the circle.

3. A diameter bisects any chord parallel to the tangent at either extremity.

If two circles have the same center, they are said to be **concentric** (see Fig. 278, p. 198).

4. In two concentric circles a chord of the larger which is a tangent of the inner is bisected at the point of tangency.

5. AB is a chord of any $\odot O$ and X is its mid-point. Prove that AB is tangent to a circle drawn with O as center and OX as radius.

6. In Fig. 287, AB is any chord and C the mid-point of $\overset{\frown}{AB}$. CD is drawn through $C \parallel AB$. Prove CD tangent to the circle.

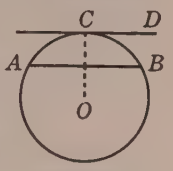


Fig. 287

†*To the teacher:* This corollary is a prerequisite for tangent circles, p. 211. The subject of tangent circles is a prerequisite for circles and loci, p. 215.

Problem 11

230. To inscribe a circle in a given triangle.

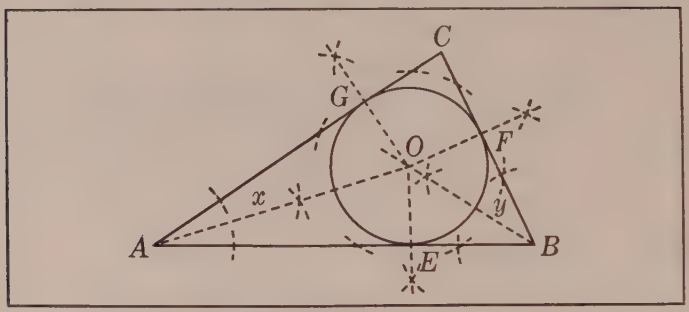


Fig. 288

Given $\triangle ABC$.

To inscribe a circle in $\triangle ABC$.

Analysis:

1. To construct a circle inscribed in $\triangle ABC$, construct a circle tangent to the sides of $\triangle ABC$.
2. $\therefore$ find a point, such as O, so situated that $\perp$s from O to the sides are radii (see Th. 36, p. 182).

Directions:

1. Draw x and y bisecting $\angle A$ and $\angle B$ respectively.
2. Let x and y intersect at O.
3. From O draw $OE \perp AB$.
4. With OE as radius and O as center draw the circle.

Analysis for the proof:

1. To prove $\odot O$ inscribed in $\triangle ABC$, prove it tangent to the sides.
2. $\therefore$ draw $OG \perp AC$ from O and $OF \perp BC$ from O and prove OG, OF, and OE radii.
3. $\therefore$ prove $OG = OF = OE$.

Note. Assume statement 1*a* in the proof if Th. 38, p. 187, was omitted.

Outline of proof. Give the reasons.

1. *a.* x and y meet at O.
   b. O lies in x.
   c. $\therefore OG = OE$.
   d. O lies in y.
   e. $\therefore OE = OF$.
   f. $\therefore OG = OE = OF$.
2. *a.* $\therefore OG$, OE, and OF are radii.
   b. OG, OE, and OF are $\perp$ the sides.
3. $\therefore$ the sides are tangent to the circle.
4. $\therefore$ the circle is inscribed in the triangle.

B Exercises

1. A tangent to a circle at the mid-point of an arc is parallel to the chord of the arc.

2. If a tangent and a chord are parallel, they cut off equal arcs.

3. If two circles have the same center, the chords of the larger which are tangents of the inner are equal.

4. Construct a circle of given radius tangent to a given line at a given point.

5. Construct a line that shall be tangent to a given circle and parallel to a given line.

6. Construct a line that shall be tangent to a given circle and perpendicular to a given line.

7. Construct a line that shall be tangent to a given circle and make a given angle with a given line.

8. Construct a circle that shall be tangent to one side of a triangle and to two sides extended.

NOTE. Such a circle is said to be **escribed** to the triangle.

Theorem 53

231. If two tangents meet at a point without a circle, the distances from the intersection to the points of tangency are equal.

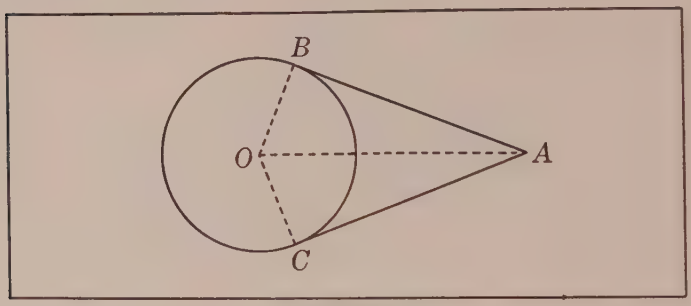


Fig. 289

Hypothesis: AB and AC are tangents to the $\odot$ at B and C respectively and meet at point A.

Conclusion: $AB = AC$.

Analysis and construction:

1. To prove $AB = AC$, prove them corresponding sides of congruent triangles.
2. $\therefore$ draw OB, OC, OA and prove $\triangle AOB \cong \triangle AOC$.

Ex. 1. If two tangents meet without a circle and radii are drawn to the points of contact, the angle made by the radii is the supplement of the angle made by the tangents.

Ex. 2. If two tangents meet without a circle, the line passing through the intersection of the tangents and the center of the circle bisects the angle made by the tangents and is the perpendicular bisector of the chord joining the points of contact.

Ex. 3. A circle can be circumscribed about a rectangle. Can a circle be circumscribed about any parallelogram? Why?

THEOREM 54 (*Optional*)

232. If two circles meet at a point on their line of centers, the circles are tangent.

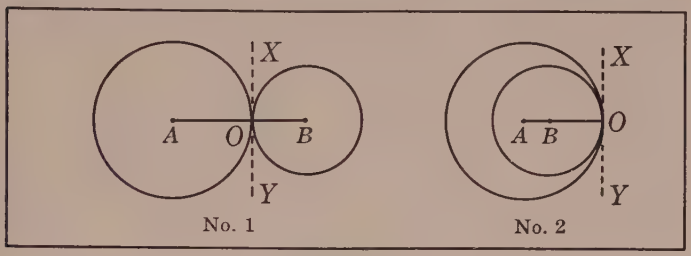


FIG. 290

Hypothesis: $\odot A$ and $\odot B$ meet at point O on the line of centers AB.

Conclusion: $\odot A$ and B are tangent.

Analysis and construction:

1. To prove $\odot A$ and $\odot B$ tangent, prove them tangent to the same line at the same point.
2. $\therefore$ draw $XY \perp AB$ at O and prove XY tangent to $\odot A$ and to $\odot B$ at O.

233. COR. I. (*Optional.*) **If two circles meet on the line of centers, the meeting point is the point of contact of the two circles.**

234. COR. II. (*Optional.*) **If the segment joining the centers of two circles is equal to the sum of the radii, the circles are tangent externally** (Fig. 290, No. 1).

235. COR. III. (*Optional.*) **If the segment joining the centers of two circles is equal to the difference between the radii, the circles are tangent internally** (Fig. 290, No. 2).

Exercise. Show how to construct circles tangent to each of two concentric circles. What are the radii?

THEOREM 55 (*Optional*)

236. If two circles are tangent, the point of contact is on the line of centers.

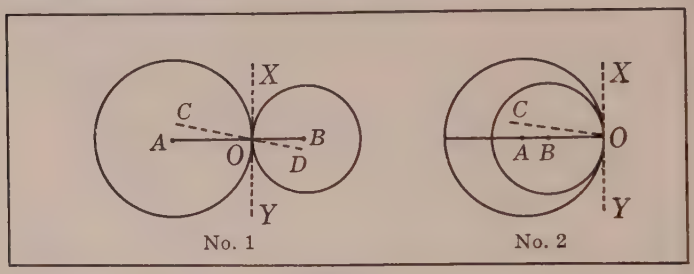


FIG. 291

Hypothesis: $\odot A$ and $\odot B$ are tangent at O. XY is the common tangent. AB is the line of centers.

Conclusion: AB passes through point O.

Analysis and construction:

1. To prove AB passes through O, prove that AB coincides with a line that does pass through O.
2. $\therefore$ draw $OC \perp XY$ at the point of contact O and prove that OC coincides with AB.
3. $\therefore$ prove that OC passes through A and B (§229, p. 207).

B Exercises

NOTE TO THE TEACHER. The exercises marked with a double dagger (‡) depend upon optional theorems.

1. In Fig. 292, XY and XZ are tangent to $\odot A$ and B respectively from any point in the common tangent OX. Prove that $XY = XZ$ if Y and Z are the points of contact.

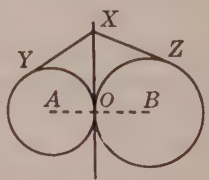


FIG. 292

‡2. Construct a circle of given radius that shall be tangent to a given circle at a given point.

‡3. If two congruent circles are tangent, the common tangent is the perpendicular bisector of the segment joining the centers.

‡4. If two congruent circles intersect, any point in the common chord or the common chord extended may be the center of a circle tangent to each of the given circles.

FIG. 293

‡5. Prove Ex. 4 for two equal tangent circles.

‡6. Prove Ex. 4 for two equal non-intersecting circles.

7. Find in Fig. 293 two circles so situated that the segment joining the centers is (1) less than the sum of the radii but greater than their difference; (2) less than the difference of the radii but greater than zero.

8. What is the relation of the segment joining the centers and the sum or difference of the radii in each of the cases shown in Fig. 293?

A Exercises

NOTE TO THE TEACHER. The exercises marked with a double dagger (‡) depend upon optional theorems.

1. Inscribe a circle in a given rhombus (see p. 135, C Ex. 4).

2. Fig. 294 shows two isosceles triangles on opposite sides of the same base. Such a figure is sometimes called a *kite*. Show that a circle can be inscribed in the kite.

Suggestion. Prove DB bisects $\angle D$ and $\angle B$. Draw the bisector of $\angle A$.

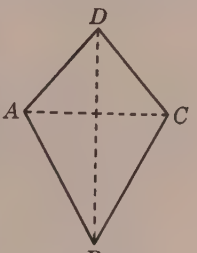


FIG. 294

3. Construct in full Fig. 295.

FIG. 295

4. A circle can be inscribed in any figure in which the bisectors of the angles meet in a point.

A **sector of a circle** is a figure bounded by two radii and the intercepted arc.

‡5. Inscribe a circle in a sector of a given circle.

‡6. Fig. 296 shows a decorated rafter design. $\angle ACB$ is a right angle. $\overset{\frown}{ADB}$ is tangent to the sides of $\angle C$ so that $AC = CB$. Construct the small circle tangent to the sides of $\angle C$ and to $\overset{\frown}{ADB}$.

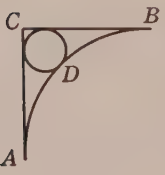


FIG. 296

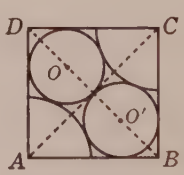


FIG. 297

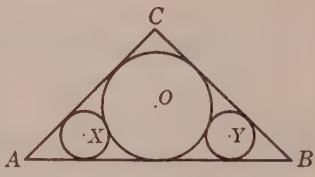


FIG. 298

‡7. Fig. 297 shows a decorated tile design. $ABCD$ is a square with its diagonals. Construct ⊙O and O' inscribed in the △ADC and ABC respectively. Construct the quadrants with A and C as centers tangent to ⊙O and O'.

NOTE. The possibilities of this figure as a design unit may be seen by drawing several figures like Fig. 297, of the same size, and placing them together in various positions.

‡8. Fig. 298 shows a window and rafter design. ABC is an isosceles right triangle. Construct ⊙O inscribed in △ABC and ⊙X and Y tangent to the sides of the triangle and to ⊙O.

‡9. Inscribe a trefoil in a given circle (Fig. 299).

NOTE. The three small circles are tangent to the large circle and to each other.

FIG. 299

TOPIC 2. CIRCLES AND LOCI (*Optional*)

237. Review the following points in Unit III, page 176:

1. State the definition of a locus (§184, p. 178, and §186, p. 179).

2. How may an unknown locus be found (§183, p. 177)?

3. State the theorems concerning the five elementary loci (§185, pp. 178 and 179).

4. What constitutes a complete proof of a locus theorem (§187, p. 179)?

5. Work at least two review exercises concerning intersecting loci (see B Exercises, p. 185).

Find a point that shall be at a given distance from a given point O and at the same time be

Ex. 1. Equally distant from two given points.

Ex. 2. Equally distant from two given parallel lines.

Ex. 3. Equally distant from two given intersecting lines.

238. Loci of centers of circles. In finding loci of centers of circles, you may think of a circle as changing in size or position or both. Fig. 300 shows a number of circles of the same radius tangent to line l. You might consider these circles as various positions of one circle that rolls or slides along the line. In this case the circle changes its position only.

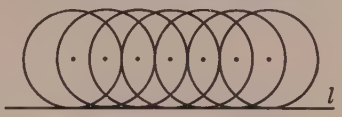


FIG. 300

In Fig. 301 six circles are shown tangent to the sides of $\angle A$; but they might be considered as various positions of one circle which expands and contracts as it rolls or slides between AB and AC. In this case the circle changes both its size and position.

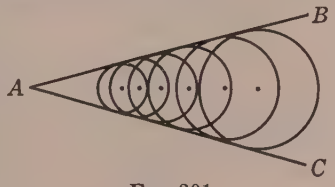


FIG. 301

Find the locus called for in each of the following exercises and give complete proof for each:

Ex. 1. Find the locus of the centers of circles which are tangent to the sides of an angle.

Analysis:

I. To find the locus, sketch in a number of circles tangent to the sides of $\angle BAC$. Connect their centers. Let AX be the locus. How is it located?

II. To prove that AX is the required locus, prove that

a. If a point is in AX, it may be the center of a circle tangent to AB and AC, the sides of $\angle A$.

b. If a point is the center of a circle tangent to AB and AC, it lies in AX.

Discussion: AX may be considered the path of the center of a circle that expands and contracts as it rolls or slides between AB and AC.

Find the locus of the center of a circle which

Ex. 2. Is tangent to each of two intersecting lines.

Ex. 3. Passes through two given points.

Ex. 4. Is tangent to a given line at a given point.

Ex. 5. Is tangent to a given line and has a given radius.

Ex. 6. Is tangent to a given circle at a given point.

Ex. 7. Is tangent to a given circle and has a given radius.

Ex. 8. Passes through a given point and has a given radius.

Ex. 9. Is tangent to each of two parallel lines.

239. Construction of circles. Make the constructions called for in the following exercises. The center of the required circle is the intersection of two loci. The discussion is often interesting.

Ex. 1. Construct a circle that shall be tangent to a given line at a given point and pass through a second given point.

Ex. 2. Construct a circle tangent to the sides of an angle and to one of these sides at a given point.

Ex. 3. Construct a circle that shall be tangent to a given circle at a given point and pass through a second given point.

Ex. 4. Construct a circle that shall be tangent to two given parallels and pass through a given point when the point is (*a*) on one of the parallels, (*b*) between the parallels.

Construct a circle of given radius that shall

Ex. 5. Be tangent to each of two given intersecting lines.

Ex. 6. Pass through a given point and be tangent to a given line or to a given circle.

Ex. 7. Be tangent to a given circle and a given line.

A Exercises

1. What is the locus of the mid-points of all equal chords of a circle?

2. Find the locus of the mid-points of segments drawn to the circle from a fixed point (*a*) without the circle, (*b*) on the circle, (*c*) within the circle. No proof is required.

3. Find the locus of the extremities of tangents to a circle if the tangents have the same length. No proof is required.

4. Find the locus of points from which tangents to a given circle meet at a given angle. No proof is required.

5. Show by a complete drawing how to construct Fig. 302. Two diagonal streets meet at A. The corner building has a front that is an arc of a circle of a given radius tangent to each of the streets. Show how to find the center of the circle for the arc BC. Show that $\triangle ABC$ is isosceles. If O is the center of the circle for the arc BC, find the number of degrees in $\angle BOC$ if $\angle A$ is 60°. What is the relation between $\angle BOC$ and $\angle A$? Why?

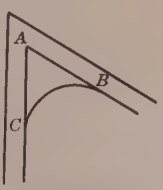


FIG. 302

6. Find the locus of points of contact of tangents drawn from a fixed point to a system of concentric circles. No proof is required.

FIG. 303

7. Construct a series of circles tangent to each other at the same point (Fig. 303). Find the locus of points of contact of tangents drawn to these circles from any point in the common tangent.

8. Make the complete drawing for the molding shown in Fig. 304. The arc AO is tangent to line AX at A. The arc BO is tangent to line BY at B. Both arcs pass through O, the midpoint of segment AB. Are the arcs AO and BO tangent to each other? Why?

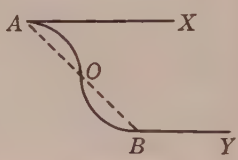


FIG. 304

NOTE. Compound curves may be made from tangent circles as shown in Fig. 305. These may be used for moldings as shown in Fig. 304, or for other architectural details.

FIG. 305

9. Find the locus of centers of circles tangent to each of two concentric circles. Can you give proof?

10. If two equal circles intersect, the common chord and the chord extended is part of the locus of centers of circles tangent to each of the two given circles. Why only a part of the locus?

11. If two equal circles are tangent, find the locus of centers of circles tangent to each of the two given circles. How many sets of tangent circles are to be considered?

12. What would be the locus mentioned in Exs. 10 and 11 if the equal circles neither intersect nor are tangent? Can you give proof?

13. Show how one of the units used in Fig. 306 is constructed. Are any loci mentioned in Exs. 9, 10, and 11 used?

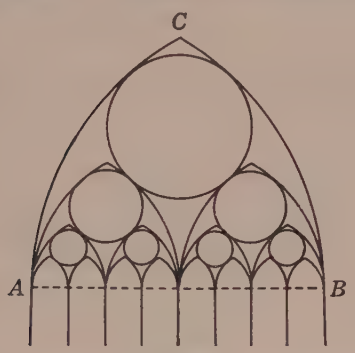


FIG. 306

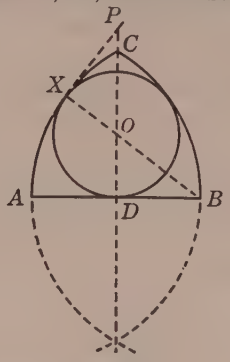


FIG. 307

NOTE. Figs. 306, 307, and 308 are church window outlines.

14. In Fig. 307, $\overset{\frown}{CA}$ and $\overset{\frown}{CB}$ are arcs of intersecting circles drawn with B and A respectively as centers. Construct a circle tangent to $\overset{\frown}{CA}$, $\overset{\frown}{CB}$, and to AB. Study the dotted lines.

15. In Fig. 308, $\triangle ABC$ is equilateral. $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$ are drawn with $\overline{AB}$ as radius and C, A, and B as centers respectively. Construct $\odot O$ tangent to $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$.

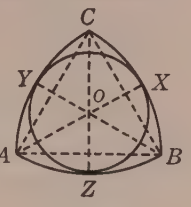


FIG. 308

Find by experiment the loci called for in Exs. 16, 17, and 18 (no proofs are required).

16. Find the locus of centers of circles tangent to each of two given circles when (*a*) one circle is within the other, (*b*) the circles intersect, (*c*) one circle is without the other, (*d*) the circles are tangent internally, (*e*) are tangent externally.

17. Find the locus of centers of circles which pass through a given point and are tangent to a given line (two cases).

18. Find the locus of centers of circles which are tangent to a given circle and to a given line (three cases).

Survey Test

1. Define circle, radius, diameter, arc, chord of a circle, chord of an arc, minor arc, major arc, congruent circles, and semicircle.

2. What can you say (*a*) of the radii of congruent circles? (*b*) of the diameter of any circle?

3. (*a*) How many arcs has every chord? (*b*) What are these arcs called? (*c*) What do these names mean? (*d*) For what special case is your answer to (*b*) not true?

4. (*a*) How many chords has every arc of a circle? (*b*) What is the chord of a semicircle?

5. (*a*) What is an angle with its vertex at the center of a circle called? (*b*) What is meant by the intercepted arc of an angle?

6. (*a*) Quote a postulate that could be used to prove central angles equal. (*b*) Quote a theorem in the proof of which this postulate was used. (*c*) Prove it.

7. (*a*) Quote a postulate that could be used to prove arcs equal. (*b*) Quote a theorem in the proof of which this postulate was used. (*c*) Prove it.

8. If a line passes through a point within a circle, in how many points do the line and circle intersect?

9. (*a*) Quote a postulate that can be used to prove that a segment extends without a circle. (*b*) Quote a theorem in the proof of which this postulate was used. (*c*) Prove it.

10. (*a*) Quote a postulate that can be used to prove that a segment lies wholly within a circle. (*b*) Quote a theorem in the proof of which this postulate was used. (*c*) Prove it.

11. Quote the theorems and postulates that could be used to prove (*a*) central angles equal; (*b*) minor arcs equal; (*c*) chords equal.

12. When have two circles a common chord? Illustrate by a drawing. What is the line of centers of two circles?

13. When would the common chord of two circles have equal arcs in these circles? Why?

14. (*a*) Does every line from the center of a circle perpendicular to a chord of that circle bisect the chord? (*b*) Quote the theorem in full.

15. A circle is to be drawn passing through three given points. (*a*) How do you determine its center? (*b*) On what does the proof depend? Quote in full. (*c*) How do you determine its radius?

16. In how many points can two circles intersect?

17. Name two ways in which a circle can be located definitely.

18. Define (*a*) a circumscribed circle; (*b*) an inscribed polygon.

19. Define (*a*) a tangent to a circle; (*b*) the point of tangency. (*c*) What other name has the point of tangency?

20. (*a*) Quote two authorities used to prove that a line is tangent to a circle. (*b*) Which is used the more often?

21. Define (*a*) an inscribed circle; (*b*) a circumscribed polygon.

22. (*a*) When are two circles tangent internally? externally? (*b*) What do you understand by the point of contact of two circles?

23. A circle is to be inscribed in a triangle. How do you determine (*a*) its center? (*b*) its radius?

24. Quote a theorem concerning circles that is used to prove lines perpendicular.

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	45, 192, 193	9	193, 205	17	198
2	192	10	193, 206	18	198
3	192, 193	11	193, 194, 195, 199	19	204
4	192	12	194	20	204, 205
5	193	13	195	21	204
6	193, 195	14	196	22	204
7	193, 195	15	197	23	208
8	193	16	198	24	206

TOPIC 3. CIRCLES AND RELATED ANGLES

240. Arcs and angles may both be measured by units called degrees.

The unit angle or **an angle of one degree** is $\frac{1}{360}$ of a complete rotation. What is meant by the **measure number** of an angle (see §49, p. 24)?

The unit arc or **an arc of one degree** is $\frac{1}{360}$ of a circle. What would be meant by the **measure number** of an arc?

Since in the same circle equal central angles intercept equal minor arcs, $\frac{1}{360}$ of a complete rotation about any point as O will intercept $\frac{1}{360}$ of any circle drawn with O as a center; that is, the unit arc is obtained directly from the unit angle. Smaller units of arc are obtained from smaller divisions of the angle. An arc of one minute (1′) corresponds to a central angle of one minute; an arc of one second (1″) corresponds to a central angle of one second.

It follows that a central angle of 30° intercepts an arc of 30°; a central angle of 42° 7′ 15″ intercepts an arc of 42° 7′ 15″. It will be accordingly assumed that if the measure of a central angle is any number whatsoever, the measure of the intercepted arc is expressed by the same number.

HISTORICAL NOTE. The use of $\frac{1}{360}$ of a circle as a degree of arc is thought to have originated with the ancient Babylonians. They were very much interested in the study of the stars. They noticed that the stars moved in circular paths and that each star returned to the same place in the heavens in 360 days. They missed the exact time by about 5 days. It is thought that this was the origin of their system of numbers based on the number 60. It was quite natural for them to take the daily change in the position of a star as a unit for measuring the movements of the stars. In a day the star would move $\frac{1}{360}$ of a circle. This is thought to be the origin of the degree of arc. All early astronomers used arcs and chords of arcs rather than central angles. Historically the degree of angle may have come from the degree of arc.

Ex. 1. Are two angles of the same number of degrees always congruent? Are two arcs of the same number of degrees always congruent? When are they congruent? Illustrate by a figure.

Ex. 2. If a wheel makes 250 revolutions a minute, through how many degrees does it revolve in one second?

241. *Post. 42. The measure number of a central angle and the measure number of its intercepted arc are the same, or a central angle is measured by its arc.*

You learned in §50, page 24, that a protractor is an instrument for measuring angles.

Post. 42 states the principle that is the basis of the use of the protractor. The division on the scale of a protractor cannot be made by geometric methods used in this book. They are made by a machine called a dividing engine.

Ex. 1. Show how to use the protractor to find the number of degrees in an arc of any circle.

Ex. 2. With a protractor construct on a given circle arcs of 40°; 75°; 135°.

Ex. 3. With a protractor divide a circle into ten equal parts. Join the points of division, thus forming an inscribed 10-side.

242. An angle is said to be **inscribed in a circle** if its vertex is on the circle and its sides are chords of the circle.

The arc cut off between the sides of an inscribed angle is called its **intercepted arc.**

Exercise. Inscribe in a given circle angles of 44°, 72°, 105°, etc. Find the measure of the intercepted arcs. Use a protractor.

A line that cuts a circle twice is called a **secant.**

THEOREM 56*

243. An inscribed angle has half as many degrees as its intercepted arc, or an inscribed angle is measured by half its intercepted arc.

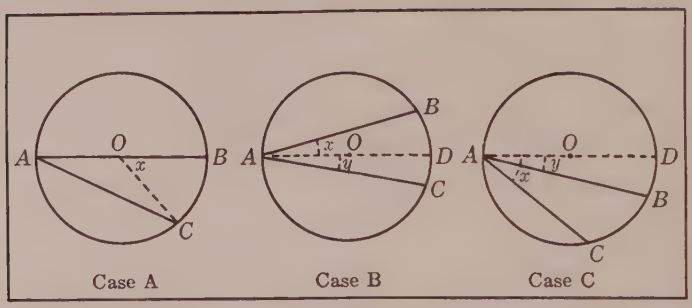


FIG. 309

Hypothesis: $\angle CAB$ is inscribed in $\odot O$.

Conclusion: The number of degrees in $\angle CAB$ is $\frac{1}{2}$ the number of degrees in $\overset{\frown}{BC}$.

Case A. When the center of the circle is on one side of the angle.

Analysis and construction:

1. To prove $\angle A$ has half as many degrees as $\overset{\frown}{BC}$, prove $\angle A$ equal to $\frac{1}{2}$ a central angle intercepted by $\overset{\frown}{BC}$.
2. $\therefore$ draw CO and prove $\angle A = \frac{1}{2}\angle x$.

Outline of proof. Give the reasons.

1. *a.* $\triangle AOC$ is isosceles.
   b. $\therefore \angle A = \angle C$.
   c. $\angle A + \angle C = \angle x$.
   d. $\therefore 2\angle A = \angle x$.
   e. $\therefore \angle A = \frac{1}{2}\angle x$.
2. *a.* $\angle x$ has as many degrees as $\overset{\frown}{BC}$.
   b. $\therefore \angle A$ has half as many degrees as $\overset{\frown}{BC}$.

Case B. When the center of circle is within the angle

Analysis: To prove $\angle A$ has half as many degrees as $\overset{\frown}{BC}$, draw diameter AD, apply Case A to $\angle x$ and $\angle y$, and add.

Outline of proof. Give the reasons.

1. $\angle x$ has half as many degrees as $\overset{\frown}{BD}$.
2. $\angle y$ has half as many degrees as $\overset{\frown}{DC}$.
3. $\therefore \angle A$ has half as many degrees as $\overset{\frown}{BC}$ (Ax. 1, p. 29).

Case C. When the center of circle is without the angle.

Analysis: To prove $\angle A$ has half as many degrees as $\overset{\frown}{BC}$, draw the diameter AD, apply Case A to $\angle x$ and $\angle y$, and subtract.

244. Cor. I. **In the same circle or in congruent circles inscribed angles measured by the same or by equal arcs are equal.**

245. Cor. II. **In the same circle or in congruent circles arcs that measure equal inscribed angles are equal.**

Show how Figs. 310 and 311 may be used to illustrate Cors. I and II.

Fig. 310

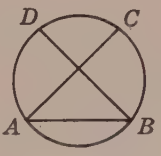


Fig. 311

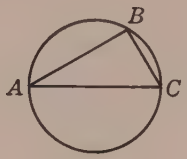


Fig. 312

246. An angle is said to be **inscribed in an arc** if its vertex lies on the arc and its sides pass through the extremities of the chord of the arc. In this case the arc is said to **contain** the angle.

247. Cor. III. **An angle inscribed in a semicircle is a right angle.**

Suggestion. In Fig. 312, which is the angle inscribed in a semicircle? By what arc is it measured?

C Exercises

1. Turn to figures elsewhere in the book and find illustrations of inscribed angles and show how each is measured in the circle. Can you find any illustrations of Cor. III, p. 225?

2. If two angles of an inscribed triangle are 55° and 72°, find the number of degrees in the arcs subtended by each side.

3. A triangle is inscribed in a circle. Two of the sides subtend arcs of 136° and 88°. Find the number of degrees in each angle of the triangle.

4. The bisector of an inscribed angle bisects its arc.

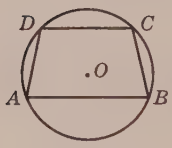


FIG. 313

FIG. 314

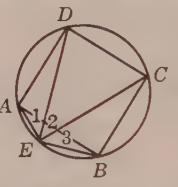


FIG. 315

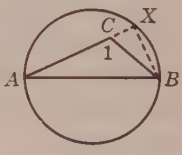


FIG. 316

5. In Fig. 313, $\overset{\frown}{AB} = 140°$ and $\overset{\frown}{DC} = 90°$. $\overset{\frown}{AD} = \overset{\frown}{BC}$. Find the number of degrees in each angle of the figure.

6. In Fig. 314, the semicircle is divided into 5 equal parts by the points C, D, F, and G. E is the mid-point of arc FD. Find the number of degrees in each angle of the figure.

7. In Fig. 315, the vertices of the square $ABCD$ lie on a circle and E is any point in the arc AB. Prove that CE and DE trisect $\angle AEB$. Find the number of degrees in $\angle 1$.

To prove $\angle AEB$ trisected, prove $\angle 1 = \angle 2 = \angle 3$.

" " $\angle 1 = \angle 2 = \angle 3$, " $\overset{\frown}{AD} = \overset{\frown}{DC} = \overset{\frown}{CB}$.

8. The angle between the segments joining a point within a circle to the ends of a diameter is an obtuse angle (Fig. 316).

To prove $\angle 1$ an obtuse angle, prove $\angle 1 >$ a rt. $\angle$.

$\therefore$ extend AC to meet the $\odot$ at X. Draw XB. Use Th. 6, p. 83, and §247, p. 225.

9. The angle between the segments joining a point without a circle to the ends of a diameter is an acute angle.

THEOREM 57

248. An angle formed by two chords intersecting within a circle is measured by one-half the sum of the intercepted arcs.

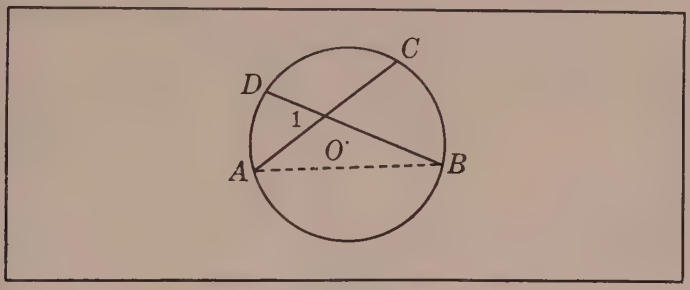


FIG. 317

Hypothesis: Chords AC and BD intersect within $\odot O$.

Conclusion: $\angle 1$ is measured by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{BC})$.

Analysis and construction:

1. To prove $\angle 1$ measured by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{BC})$, prove $\angle 1$ equal to the sum of two angles that are measured by $\frac{1}{2}\overset{\frown}{AD}$ and $\frac{1}{2}\overset{\frown}{BC}$ respectively.
2. $\therefore$ draw AB and prove $\angle 1 = \angle B + \angle A$.

Outline of proof. Give the reasons.

1. $\angle 1 = \angle B + \angle A$.
2. $\angle B$ has $\frac{1}{2}$ as many degrees as $\overset{\frown}{AD}$.
3. $\angle A$ has $\frac{1}{2}$ as many degrees as $\overset{\frown}{BC}$.
4. $\therefore \angle 1$ is measured by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{BC})$.

Ex. 1. If, in Fig. 318, $\overset{\frown}{CD} = 55°$, $\overset{\frown}{AB} = 140°$, and $\overset{\frown}{BC} = 66°$, find the number of degrees in each angle of the figure.

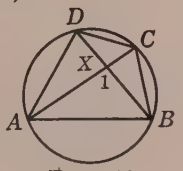


FIG. 318

Ex. 2. Answer Ex. 1 if $\angle ACB = 50°$, $\angle DCA = 42°$, and $\angle CBD = 31°$.

THEOREM 58

249. An angle formed by a tangent and a chord is measured by one-half its intercepted arc.

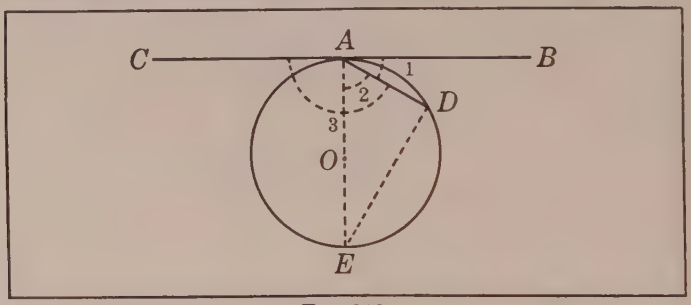


FIG. 319

Hypothesis: AB is tangent to $\odot O$ at A. $\overline{AD}$ is drawn from A, the point of contact of the tangent.

Conclusion:
A. $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.
B. $\angle 3$ is measured by $\frac{1}{2}\overset{\frown}{AED}$.

A. Analysis and construction:

1. To prove $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AD}$, prove $\angle 1$ equal to an angle that is measured by $\frac{1}{2}\overset{\frown}{AD}$.
2. $\therefore$ draw the diameter AE and prove $\angle 1 = \angle E$.

Outline of proof. Give the reasons.

1. *a.* $\triangle ADE$ is a rt. $\triangle$.
   b. $\therefore \angle 2 + \angle E = 1$ rt. $\angle$.
   c. $\angle 1 + \angle 2 = 1$ rt. $\angle$.
   d. $\therefore \angle 1 = \angle E$.
2. *a.* $\angle E$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.
   b. $\therefore \angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.

B. *Suggestion.* Prove $\angle 3 = \angle 2 + \angle D$ and add the arcs that measure $\angle 2$ and $\angle D$.

250. Cors. I and II, page 225, are for inscribed angles only. The following postulates are more general.

Post. 43. In the same circle or in congruent circles, angles are equal if they are measured by equal arcs.

Post. 44. In the same circle or in congruent circles, arcs are equal if they measure equal angles.

C Exercises

1. Turn to figures elsewhere in the book and find illustrations of angles formed by a tangent and a chord and of angles formed by two chords intersecting within a circle. Show how each angle is measured in the circle.

2. In Fig. 320, the sides of $\triangle XYZ$ are tangent to the circle at the vertices of the inscribed triangle ABC. If $\angle ACB = 71°$ and $\angle CBA = 58°$, find the number of degrees in each angle of the figure.

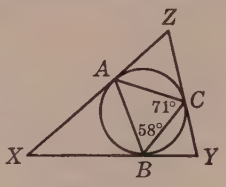


FIG. 320

3. A chord forms equal angles with the tangents at its extremities.

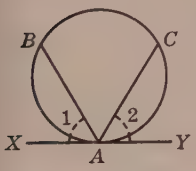


FIG. 321

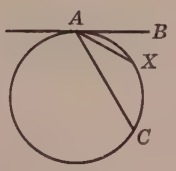


FIG. 322

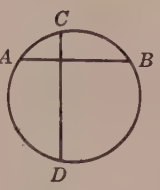


FIG. 323

FIG. 324

4. In Fig. 321, XY is tangent to the circle. If $\angle 1 = \angle 2$, prove $\overline{AB} = \overline{AC}$.

5. In Fig. 321, XY is tangent to the circle. If $\overline{AB} = \overline{AC}$, prove $\angle 1 = \angle 2$.

6. In Fig. 322, AB is tangent to the circle. If AX bisects $\angle BAC$, prove X the mid-point of $\overset{\frown}{AC}$.

7. State and prove the converse to Ex. 6.

8. In Fig. 323, $\overline{AB} \perp \overline{CD}$. Prove $\overset{\frown}{AC} + \overset{\frown}{BD}$ = a semicircle.

9. In Fig. 324, $\overline{AX} = \overline{BY}$. Prove $\triangle ABC$ isosceles.

10. State and prove the converse to Ex. 9.

THEOREM 59

251. An angle formed by (A) two secants, (B) a secant and a tangent, or (C) two tangents intersecting without a circle is measured by one-half the difference of the intercepted arcs.

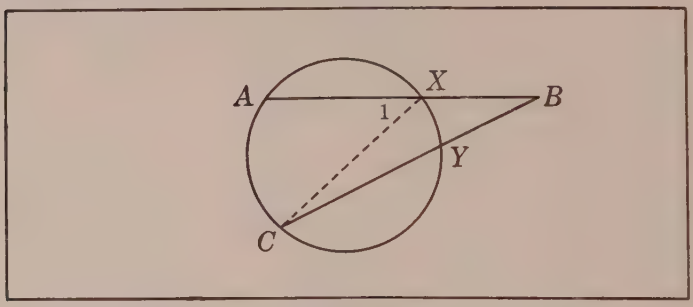


FIG. 325

Case A:

Hypothesis: Secants BA and BC intersect without a $\odot$.

Conclusion: $\angle B$ is measured by $\frac{1}{2}(\overset{\frown}{AC}-\overset{\frown}{XY})$.

Analysis and construction:

1. To prove $\angle B$ measured by $\frac{1}{2}(\overset{\frown}{AC}-\overset{\frown}{XY})$, prove $\angle B$ equal to the difference of two angles that are measured by $\frac{1}{2}\overset{\frown}{AC}$ and $\frac{1}{2}\overset{\frown}{XY}$ respectively.
2. $\therefore$ draw CX and prove $\angle B = \angle 1 - \angle C$.
3. $\therefore$ prove $\angle B + \angle C = \angle 1$.

Outline of proof. Give the reasons.

1. $\angle B + \angle C = \angle 1$.
2. $\therefore \angle B = \angle 1 - \angle C$.
3. $\angle 1$ is measured by $\frac{1}{2}\overset{\frown}{AC}$.
4. $\angle C$ is measured by $\frac{1}{2}\overset{\frown}{XY}$.
5. $\therefore \angle B$ is measured by $\frac{1}{2}(\overset{\frown}{AC}-\overset{\frown}{XY})$.

Draw the figures for Cases B and C and prove.

THEOREM 60

252. If two chords, a chord and a tangent, or two tangents are parallel, they intercept equal arcs on a circle.

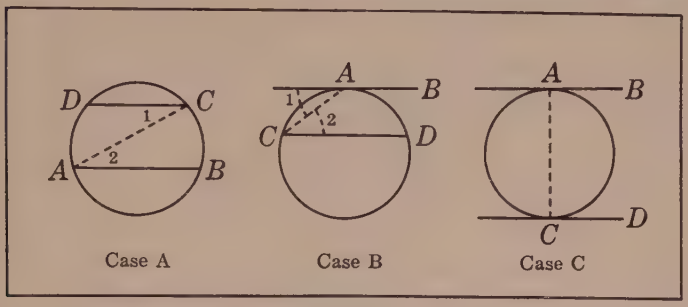


FIG. 326

Case A:

Hypothesis: $\overline{AB} \parallel \overline{CD}$ in any circle.

Conclusion: $\overset{\frown}{AD} = \overset{\frown}{BC}$.

Analysis and construction:

1. To prove $\overset{\frown}{AD} = \overset{\frown}{BC}$, prove that they measure equal angles.
2. $\therefore$ draw AC and prove $\angle 1 = \angle 2$.

Case B and Case C are left to you to prove.

Ex. 1. If two arcs on a circle are equal, the chords joining their extremities are either equal or parallel, depending upon the order in which the points are joined.

Ex. 2. (*Optional.*) Prove Th. 57, p 227, by proving $\angle 1$ equal to one inscribed angle (Fig. 327).

Suggestion. Draw DX from $D \parallel AB$. Prove $\angle 1 = \angle D$ and $\overset{\frown}{AD} + \overset{\frown}{CB} = \overset{\frown}{CX}$.

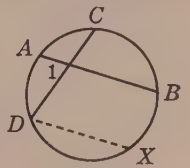


FIG. 327

Ex. 3. (*Optional.*) Prove either part of Th. 59, p. 230, by a method similar to that suggested in Ex. 2 for Th. 57.

Problem 12

253. To construct a tangent to a circle from a given point without the circle.

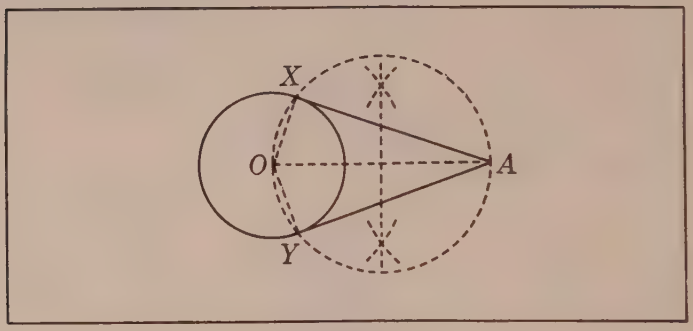


Fig. 328

Given $\odot O$ and point A without the circle.

To construct a tangent to $\odot O$ from point A.

Analysis: In order to construct a tangent to $\odot O$ from point A, construct a right angle whose vertex is on $\odot O$ and whose sides pass through points O and A.

Directions:

1. Draw OA.
2. Construct a circle on OA as diameter, cutting $\odot O$ at X and Y.
3. Draw AX and AY.

Outline of proof. Give the reasons.

1. Draw OX and OY.
2. *a.* $\angle OXA$ is a rt. $\angle$.
   b. $\therefore AX \perp$ radius OX.
   c. $\therefore AX$ is tangent to $\odot O$.
3. Similarly AY is tangent to $\odot O$.

B Exercises Concerning Tangents

1. Construct a common external tangent to two circles.

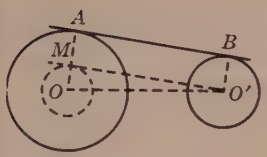


FIG. 329

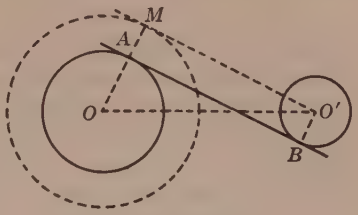


FIG. 330

Directions (Fig. 329):

I. Draw the dotted circle with O as center and the difference between the radii of $\odot O$ and $\odot O'$ as radius.
II. From O' draw $O'M$ tangent to the dotted circle at M.
III. Draw OM and extend to meet $\odot O$ at A.
IV. Draw $O'B$ from $O' \perp O'M$, meeting $\odot O'$ at B.
V. Draw AB. AB is the required tangent.

Analysis:

I. To prove AB tangent to $\odot O$ and O', prove $AB \perp OA$ and $\perp O'B$.
II. To prove $AB \perp OA$, prove $O'M \perp OA$ and $\parallel AB$.
III. " " $O'M \parallel AB$, " $O'BAM$ a $\square$.
IV. " " $O'BAM$ a $\square$, " $O'B =$ and $\parallel AM$.
V. $\therefore$ " $O'M \perp O'B$ and $\perp OA$.

2. Construct a common internal tangent to two circles. The radius of the dotted circle is equal to the sum of the radii of the two given circles (Fig. 330).

Note the application of Exs. 1 and 2 to belts over pulleys. Bring in some drawing illustrating this.

The figure for the two external tangents is used in astronomy in connection with eclipses. Draw the second external tangent and extend the two tangents until they intersect. Look this up in some text on astronomy.

3. How many common tangents can be drawn to two circles if they are in each of the possible positions shown in Fig. 293, p. 213? Show how to construct the tangents in each case.

4. Prove that a tangent can be drawn to a circle from a given point without the circle by the following method (Fig. 331): Let A be the given point. Join OA. With O as center and OA as radius draw circle ABC. Let OA cut the given circle at D. At D draw BC tangent to the given circle and cutting the outer circle at B and C. Draw OB and OC cutting the given circle at E and F. AE and AF are tangent to the given circle.

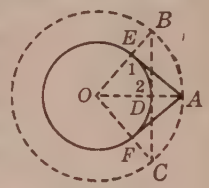


FIG. 331

Suggestion. Prove $\angle 1 = \angle 2 = 1$ rt. $\angle$; $\therefore$ prove $\triangle OEA \cong \triangle ODB$.

NOTE. The construction given in Ex. 4 is similar to Euclid's.

5. A line that is tangent to each of two equal circles is parallel to the segment joining their centers or it bisects it.

6. Two common internal tangents to two circles are equal.

7. Is the same true of the external tangents? Prove it.

B Exercises

1. Two chords intersect within a circle at an angle of 58°. If one of the intercepted arcs is 95°, find the other.

2. Two secants intersect without a circle at an angle of 38°. If the larger of the two intercepted arcs is 106°, find the other.

3. Two secants intersect without a circle at an angle of 40°. If the smaller of the two intercepted arcs is 15°, find the other.

4. In Fig. 332, if $\overset{\frown}{AXC}$ is 220°, find the number of degrees in $\angle B$. BA and BC are tangent to the circle.

5. Two tangents are perpendicular to each other. How many degrees in the arcs formed by the points of tangency?

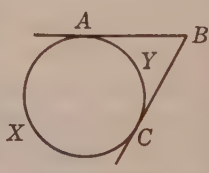


FIG. 332

6. In Fig. 332, if $\angle B$ is 68°, find the number of degrees in $\overset{\frown}{AXC}$ and $\overset{\frown}{AYC}$.

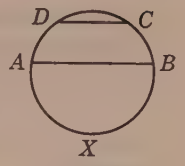


FIG. 333

7. If, in Fig. 333, $AB \parallel CD$, $\overset{\frown}{AD} = 32°$, and $\overset{\frown}{DC}$ is one-third of $\overset{\frown}{AXB}$, find the number of degrees in each arc.

8. If, in Fig. 334, $\angle C = 25°$ and $\overset{\frown}{XY} = \frac{2}{3}\overset{\frown}{AB}$, find the number of degrees in $\angle XAY$ and in $\angle BYA$.

9. If $\angle C = 24°$, $\overset{\frown}{XY} = \frac{2}{5}\overset{\frown}{AB}$, and $\overline{AY}$ is a diameter, find the number of degrees in each angle of the figure in Fig. 334.

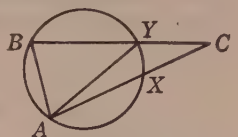


FIG. 334

10. If, in Fig. 335, $\overset{\frown}{DC} = 55°$, $\overset{\frown}{AB} = 140°$, and $\overset{\frown}{BC} = \frac{2}{3}\overset{\frown}{AD}$, find the number of degrees in each angle of the figure.

11. When will the perpendicular bisectors of the sides of a triangle meet on one of the sides of the triangle? When will they meet within the triangle? When will they meet without the triangle? Why?

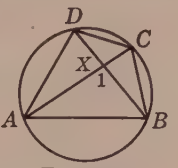


FIG. 335

12. A circle is circumscribed about $\triangle ABC$ and P is the mid-point of $\overset{\frown}{AB}$. Prove that $\angle ABP$ is $\frac{1}{2}\angle C$.

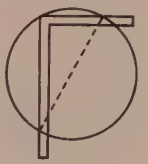


FIG. 336

Construct a right triangle, given

13. The hypotenuse and one leg.

14. The hypotenuse and an acute angle.

15. Show that the center of a given circle may be found with a carpenter's steel square as shown in Fig. 336. Give reasons.

16. Show that the center of a given arc may be found by using two carpenter's steel squares as shown in Fig. 337. Give reasons.

FIG. 337

17. ABC is a triangle inscribed in a circle whose center is O. OD is perpendicular to AC. Prove that $\angle COD = \angle B$.

18. Investigate the case in which $\angle B$ is an obtuse angle.

19. Find the arc which measures $\angle ACD$ in Fig. 338.

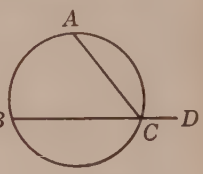


FIG. 338

20. If, from point P on a circle, $\overline{PQ}$ and a diameter PR are drawn, a radius parallel to $\overline{PQ}$ bisects $\overset{\frown}{QR}$.

21. Construct a number of angles inscribed in the same arc and the bisector of each angle. What theorem can be inferred from the drawing? Prove it.

22. If two chords intersect in a circle and a segment of one is equal to a segment of the other, the chords are equal.

23. State and investigate the converse of Ex. 22. Prove your conclusions.

24. If the radius of one circle is the diameter of a second, any chord of the larger drawn from the point of contact is bisected by the smaller circle.

25. In Fig. 339, $\odot O$ and O' are two equal circles intersecting at A and B. C and D are any points on $\odot O$ and O' respectively. Prove $\angle BCA = \angle ADB$.

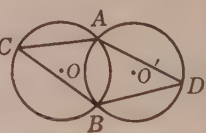


FIG. 339

26. An angle between two tangents to a circle is double the angle between the chord joining the points of contact and the radius drawn to one point of contact.

27. In Fig. 340, P is the mid-point of $\overset{\frown}{CD}$. PA and PB are any two chords from P. Prove that $\angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$.

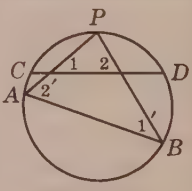


FIG. 340

28. If a tangent is drawn to a circle at the extremity of a chord, the mid-point of the intercepted arc is equally distant from the chord and the tangent.

29. The opposite angles of an inscribed quadrilateral are supplementary.

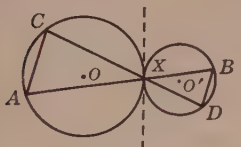


FIG. 341

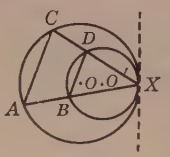


FIG. 342

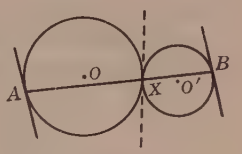


FIG. 343

30. If, in Fig. 341, ⊙O and O' are tangent at X and $\overline{AB}$ and $\overline{CD}$ pass through the point of contact, prove that AC is parallel to BD.

Suggestion. Draw the common tangent.

31. Prove Ex. 30 if the circles are tangent internally (Fig. 342).

32. If, in Fig. 343, ⊙O and O' are tangent at X and $\overline{AB}$ passes through point X, prove that tangents at A and B are parallel.

33. Prove Ex. 32 if the circles are tangent internally.

34. In Fig. 344, ⊙O and O' are tangent at X and $\overline{AB}$ passes through X. Prove that diameters from A and B are parallel.

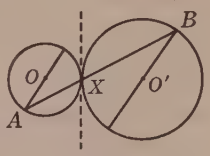


FIG. 344

35. Prove Ex. 34 if the circles are tangent internally.

36. If two equal circles intersect and a segment is drawn through either point of intersection terminating in the circles, the segments joining the extremities with the other point of intersection of the circles are equal.

37. Prove Th. 58, p. 228, by comparing $\angle CAB$ with a central angle (Fig. 345).

38. In Fig. 346, let AB and CD represent any two chords intersecting within $\odot O$. How is $\angle 1$ measured? How will the measure of $\angle 1$ change if CD revolves about point C in either direction? Discuss the case in which CD becomes parallel to AB.

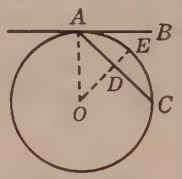


FIG. 345

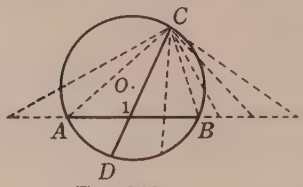


FIG. 346

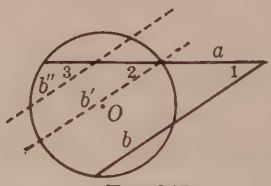


FIG. 347

39. In Fig. 347, secants a and b meet without $\odot O$. If secant b moves until it comes into the positions b' and b'', how are $\measuredangle 1$, 2, and 3 measured? Give the theorem that applies in each case.

40. Let P represent any point within $\odot O$ (Fig. 348). Suppose P to move away from the center along the ray OX. How many tangents can be drawn to $\odot O$ through point P when P is within the circle? when P is on the circle? when P is outside the circle? How are these tangents constructed in each case? Locate various positions of P outside the circle and sketch in the tangents from P at each position of P. As P moves away from the circle how does the angle between the tangents change? How do the points of contact move? What are the limiting positions of these points? How does the length of the segment between P and the point of contact change? What are the limiting lengths of this segment?

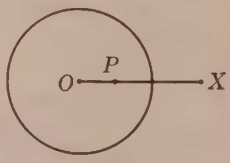


FIG. 348

PROBLEM 13 (*Optional*)

254. Upon a given segment as a chord construct an arc of a circle such that every angle inscribed in it shall equal a given angle.

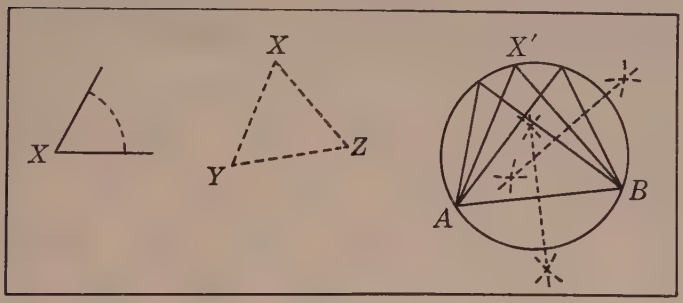


FIG. 349

Given segment AB and $\angle X$.

To construct on AB as a chord an arc such that every angle inscribed in it shall equal $\angle X$.

Analysis: To construct an arc such that all angles inscribed in it shall be equal to $\angle X$, construct an arc such that any angle inscribed in it shall equal $\angle X$.

Directions:

1. Draw $\triangle XYZ$ any triangle with one angle equal to $\angle X$.
2. On AB as base construct $\triangle ABX'$ with its base angles equal to $\angle Y$ and $\angle Z$.
3. Circumscribe a circle about $\triangle ABX'$.
4. All angles inscribed in $\overset{\frown}{AX'B}$ will equal $\angle X$.

Proof: Prove that $\angle X = \angle X'$ and that all angles inscribed in $\overset{\frown}{AX'B}$ equal $\angle X'$.

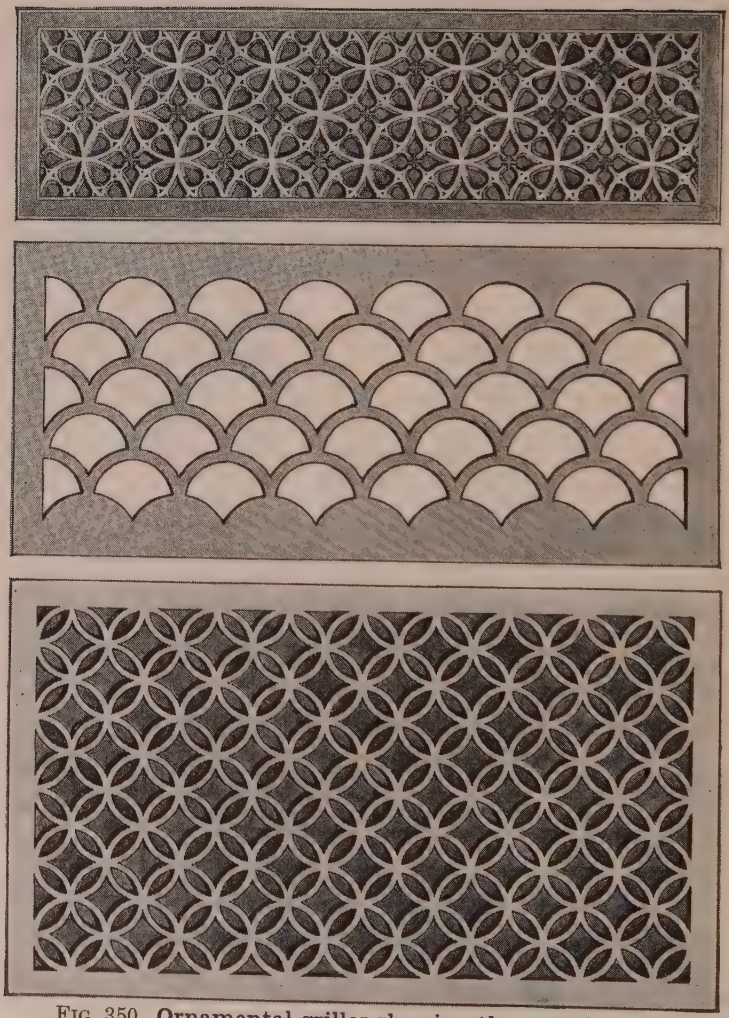


FIG. 350. **Ornamental grilles showing the use of circles**

Survey Test

1. What is meant by the measure number (*a*) of an angle? (*b*) of an arc?

2. (*a*) When is an angle said to be inscribed in a circle? (*b*) What is meant by the intercepted arc of an angle?

3. Define a secant.

4. What is meant by the statement that an angle is measured by its arc or by half its arc?

5. Reword Th. 57, p. 227, Th. 58, p. 228, and Th. 59, p. 230, so as to eliminate the words "is measured by."

6. (*a*) Quote five authorities concerning the measurement of angles by arcs. (*b*) Which would you consider fundamental and why?

7. Of the authorities quoted in answer to question 6, which do you think was used the most often in other theorems and exercises?

8. Quote four theorems the proofs of which depend upon Th. 56. Draw the figures for these theorems and show where Th. 56 applies in each case.

9. Draw the figures for Ths. 56–59 and show where Th. 18, p. 106, applies in the proofs.

10. How may the theorems concerning the measurement of angles be used to prove (*a*) angles equal? (*b*) arcs equal? (*c*) lines perpendicular? Quote in full.

11. When is an angle measured by (*a*) half the difference of two arcs? (*b*) half of one arc? (*c*) half the sum of two arcs? (*d*) one whole arc?

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	222	6	223, 224	9	224, 227
2	223		227, 228, 230		228, 230
3	223	8	227, 228	10	225, 229, 231
4	223, 224		230, 231	11	230, 224, 228
					227, 223

255. SUMMARY II

Loci

A line or group of lines is called a locus if:

1. It contains all points that fulfill some given requirement, and
2. Contains no other points. . . . . §186, p. 179

To find a locus when the requirement is given, locate a great many points that fulfill the requirement. . . . . §183, p. 177

Locus theorems:

1. The locus of points which are:
   a. At a given distance from a given point. . . . . p. 179
   b. Equally distant from two given points. . p. 179; Th. 35, p. 180
   c. Equally distant from two given parallels. . . . . p. 179
   d. At a given distance from a given straight line. . . . . p. 179
   e. Equally distant from the sides of an angle p. 179; Th. 36, p. 182

‡2. Locus of centers of circles. . . . . §238, pp. 215 and 216

Use of locus theorems:

1. In locating points. . . . . C Exercises, p. 184; B Exercises, p. 185

‡2. In constructing circles. . . . . §239, p. 216, and A Exercises, p. 217

‡**Lines that meet at one point** (or **are concurrent**):

1. The perpendicular bisectors of the sides of a triangle. Th. 37, p. 186
2. The bisectors of the angles of a triangle. . . . . Th. 38, p. 187
3. The lines through the vertices of a triangle perpendicular to the opposite sides. . . . . Th. 39, p. 188
4. The medians of a triangle. . . . . Th. 30, p. 145

Circles

A circle is located definitely if:

1. Its center and radius are known. . . . . §200, p. 192; §213, p. 198
2. It passes through three given points. . . §212, p. 197; §213, p. 198

‡Optional.

Chords in the same circle or in congruent circles are equal if:

1. They have equal central angles.....Th. 40B, p. 194
2. They have equal minor arcs.....Th. 41B, p. 195
3. They are equally distant from the center.....Th. 43, p. 199

Segments (not chords) are equal if:

1. A line through the center is perpendicular to a chord.....Th. 42, p. 196
2. The line of centers of two circles intersects a common chord..... Exs. 1 and 2, §207, p. 194
3. They are perpendiculars from the center to two equal chords in the same circle or in congruent circles....Th. 44, p. 199
4. Two tangents to a circle are drawn from a point without a circle.....Th. 53, p. 210

Arcs in the same circle or in congruent circles are equal if:

1. They are semicircles.....Post. 29, p. 192; Th. 60, p. 231
2. They have equal central angles.....Post. 31, p. 193
3. They have equal chords.....Th. 41A, p. 195
4. They are intercepted by parallel chords.....Th. 60, p. 231
5. They are intercepted by a chord and a tangent parallel to it.....Th. 60, p. 231
6. A line through the center is perpendicular to the chord of an arc.....Th. 42, p. 196
7. They measure equal angles.....Post. 44, p. 229

Angles in the same circle or in congruent circles are equal if they are:

1. Central angles having equal minor arcs.....Post. 32, p. 193
2. Central angles having equal chords.....Th. 40A, p. 194
3. Measured by equal arcs.....Post. 43, p. 229

Perpendiculars and right angles are formed when:

1. The line of centers intersects the common chord of two circles.....§207, Ex., p. 194
2. An angle is inscribed in a semicircle.....§247, p. 225
3. A radius is drawn to the point of contact of a tangent.....Th. 52, p. 206

A line is tangent to a circle if:

1. It touches the circle at one point only.....§224, p. 204
2. It is perpendicular to a radius at the outer extremity of the radius.....Th. 51, p. 205

Construction of tangents.....Ex., p. 205; Prob. 12, p. 232

Two circles are tangent if:

1. They are tangent to the same line at the same point..§226, p. 204

‡2. They meet at a point on the line of centers.....Th. 54, p. 211

‡3. The segment joining the centers is equal to:
   a. The sum of the radii.....§234, p. 211
   b. The difference of the radii.....§235, p. 211

For measurement of angles note the measure of:

1. A central angle.....Post. 42, p. 223
2. An inscribed angle.....Th. 56, p. 224
3. An angle formed by a tangent and a chord.....Th. 58, p. 228
4. An angle formed by two chords intersecting within a circle.....Th. 57, p. 227
5. An angle formed by two secants, two tangents, or a secant and a tangent intersecting without a circle..Th. 59, p. 230

Construction of circles:

1. A circle circumscribed about a triangle §214, p. 198; Prob. 10, p. 197
2. A circle inscribed in a triangle.....§225, p. 204; Prob. 11, p. 208
3. A circle escribed to a triangle.....B Ex. 8, p. 209

‡4. By intersecting loci.....§239, p. 216

Construction lines. A line may be drawn:

1. From the center, perpendicular to a chord.....Th. 42, p. 196
2. A perpendicular bisector of a chord.....§210, p. 196
3. From the center to the point of contact of a tangent.Th. 52, p. 206

‡4. From the point of contact perpendicular to a tangent.§229, p. 207

1. It pas

2. For inequalities in circles see §217, p. 200, and Ths. 45–50, pp. 201–203.

‡Optional.

C Exercises—Oral Review

Complete each of the next five statements. The locus of points

1. At a given distance from a given point is . . . . . .

2. Equally distant from two given points is . . . . . .

3. At a given distance from a given straight line is . . . . . .

4. Equally distant from two given parallels is . . . . . .

5. Equally distant from the sides of an angle is . . . . . .

6. State in full the two facts that must be proved in order to prove completely the theorem referred to in each of the first five questions.

7. If you do not know what a locus is, how may you proceed to find it?

8. How would you find the locus of the vertices of right triangles having a given hypotenuse?

9. State two methods, dependent upon circles, for obtaining perpendiculars and right angles.

10. State three methods, dependent upon circles, for proving angles equal.

11. Distinguish between an angle degree and an arc degree.

12. How many circles can pass through one given point? How would you obtain their centers?

13. Answer question 12 for two given points; for three given points.

14. When may a straight line have only one point in common with a circle? two points? three points?

15. Give two ways to prove that a line is tangent to a circle. Which of these methods is obtained from a theorem concerning tangents? On what does the proof of this theorem depend? Which method is used most frequently in exercises and theorems?

16. Explain what is meant by the statement that an angle is measured by half its arc.

17. What part of a circle is intercepted by an inscribed right angle? by an inscribed obtuse angle?

18. When will the center of the circle circumscribed about a triangle lie outside the triangle? When will it lie on one side of the triangle?

19. What is a protractor? On what fact in geometry does its construction depend? How is it used to measure (*a*) angles; (*b*) arcs?

20. Are two tangents to a circle equally distant from the center? Why?

21. Quote in full all the theorems concerning the measurement of angles which intercept arcs on a circle.

22. When is an angle measured by one-half its intercepted arc? by the whole of its intercepted arc?

23. When is an angle measured by one-half the sum of its intercepted arcs? by one-half the difference of its intercepted arcs?

24. Quote a theorem in circles of which the converse is not true.

25. Show that parallel lines cut off equal arcs on a circle.

The next five exercises refer to Fig. 353, page 247:

26. If $\overset{\frown}{AE} = 110°$, how many degrees are there in $\angle BAE$? Why? In $\angle ACE$? Why?

27. If $\overset{\frown}{AE} = 110°$ and $\overset{\frown}{DE} = 25°$, how many degrees are there in $\angle EAD$? in $\angle BAD$? in $\angle ACD$? Why?

28. If $\overset{\frown}{AE} = 136°$ and $\angle BAD = 80°$, find the number of degrees in $\angle EAD$. Give reasons.

29. If $\overset{\frown}{AE} = 114°$ and $\overset{\frown}{CD} = 36°$, find the number of degrees in $\angle AXE$. Give reasons.

30. If $\angle AXE = 100°$, $\overset{\frown}{AE} = 70°$, and $\overset{\frown}{DE} = 20°$, find the number of degrees in $\overset{\frown}{CD}$ and $\overset{\frown}{CA}$. Give reasons.

B Exercises—Review on Circles

NOTE TO THE TEACHER. The following exercises are suitable for review of circles. Work marked optional is in no way involved.

1. Construct a circle of given radius that shall pass through two given points.

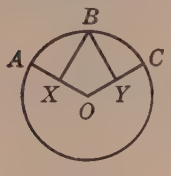


FIG. 351

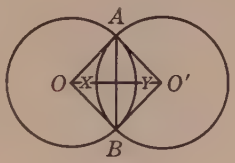


FIG. 352

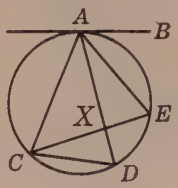


FIG. 353

2. In Fig. 351, B is the mid-point of $\overset{\frown}{AC}$. BX and BY are perpendicular to OA and OC respectively. Prove $BX = BY$.

3. If, in Fig. 351, BX and BY are perpendicular to OA and OC respectively and $BX = BY$, prove $\overset{\frown}{BA} = \overset{\frown}{BC}$.

4. If, in Fig. 352, $\odot O$ and O' are two equal intersecting circles, prove $\overset{\frown}{AXB} = \overset{\frown}{AYB}$.

5. If, in Fig. 352, $\odot O$ and O' are two equal circles intersecting so that $OBO'A$ is a square, prove $O'A$ tangent to $\odot O$ and OA tangent to $\odot O'$.

6. In Fig. 353, AB is tangent to the circle at A. $\overline{AC}$ and $\overline{AE}$ are not equal. AD is not a diameter. Find illustrations of angles that are equal because they are measured by half the same arc.

7. Show how to bisect a given arc.

8. Construct through a given point within a circle a chord that shall be bisected at the given point.

9. Construct in a given circle a chord that shall be equal to a given chord and (*a*) parallel to it; (*b*) perpendicular to it.

10. If two arcs of a circle are equal, their chords are equally distant from the center.

11. State and prove the converse to Ex. 10.

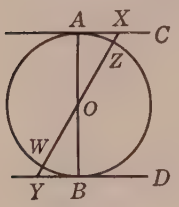


FIG. 354

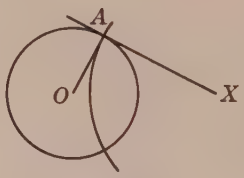


FIG. 355

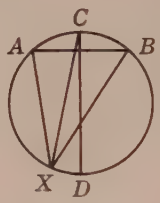


FIG. 356

12. In Fig. 354, AB is a diameter of $\odot O$. AC and BD are tangents at the ends of the diameter. XY is drawn through O to meet the tangents at X and Y and to cut the circle at Z and W. Prove $XZ = YW$.

13. In Fig. 355, AX is given tangent to $\odot O$ at A. Prove OA tangent to a circle drawn with X as a center and AX as a radius.

14. In Fig. 356, CD is a diameter $\perp \overline{AB}$. Prove that CX bisects $\angle AXB$.

15. If, in Fig. 357, diameter CD is perpendicular to $\overline{AB}$, prove $\angle 1 = \angle 2$.

16. If, in Fig. 357, CD is a diameter and $\angle 1 = \angle 2$, prove CD the perpendicular bisector of AB.

17. If, in Fig. 357, C is the mid-point of $\overset{\frown}{AB}$ and $\angle 1 = \angle 2$, prove CD a diameter of the circle.

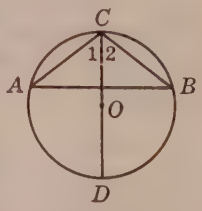


FIG. 357

18. The sides of a circumscribed polygon are equally distant from the center of the circle.

19. If a chord of a circle is equal to the radius, the central angle is 60°.

20. From Ex. 19 show how to inscribe in a circle an equilateral polygon of six sides and prove that it is regular.

21. If two tangents to a circle meet at an angle of 60°, the chord joining the points of contact is equal to the tangents.

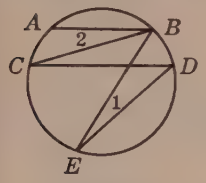


FIG. 358

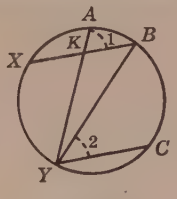


FIG. 359

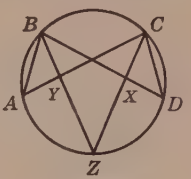


FIG. 360

22. If, in Fig. 358, $AB \parallel CD$, prove $\angle 1 = \angle 2$.

23. In Fig. 358, $\angle 2$ is any angle and $\angle 1 = \angle 2$. If C and D are joined as shown, prove $AB \parallel CD$.

24. If, in Fig. 359, $\overset{\frown}{AB} = 38°$, $\angle 1 = 66°$, and $YC \parallel XB$, find the number of degrees in $\angle 2$.

25. If, in Fig. 359, BY bisects $\angle AYC$, $XB \parallel YC$, and $\overset{\frown}{BC} = 42°$, find the number of degrees in $\angle 1$.

26. If, in Fig. 359, BY bisects $\angle AYC$ and $XB \parallel YC$, show that $\overset{\frown}{AB} = \overset{\frown}{XY}$ and that $\triangle KYB$ is isosceles.

27. In Fig. 360, if $\overset{\frown}{AB} = \overset{\frown}{CD}$ and $\overset{\frown}{AZ} = \overset{\frown}{DZ}$, prove $\triangle BAY \cong \triangle CDX$.

28. In Fig. 361, if $AD \parallel BE$, $\angle 1 = 32°$, $\overset{\frown}{CE} = 86°$, find number of degrees in $\overset{\frown}{AB}$.

29. If, in Fig. 361, C is the mid-point of $\overset{\frown}{AD}$, $BE \parallel AD$, $\angle 1 = 28°$, and $\overset{\frown}{AB} = 10°$, find the number of degrees in $\overset{\frown}{BE}$.

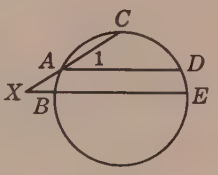


FIG. 361

30. If, in Fig. 361, $\angle X = 36°$ and $\overset{\frown}{AB} = 12°$, find the number of degrees in $\overset{\frown}{CE}$.

31. If, through two congruent circles, a straight line is drawn parallel to the line joining their centers, the chords intercepted by the two circles are equal.

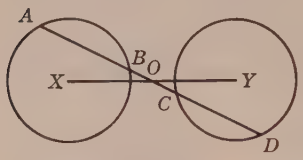


FIG. 362

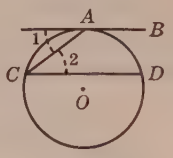


FIG. 363

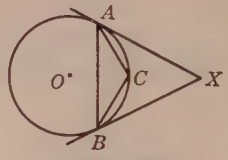


FIG. 364

32. In Fig. 362, X and Y are two congruent circles, O is the mid-point of XY, and AD is any line through O cutting both circles. Prove $\overline{AB} = \overline{CD}$.

33. If, in Fig. 363, AB is tangent to the circle and CD is drawn so that $\angle 1 = \angle 2$, prove arc AC = arc AD.

34. In Fig. 364, AX and BX are tangents to the circle and BC bisects $\angle XBA$. Prove arc BC = arc AC and that AC bisects $\angle XAB$.

35. Circumscribe about a given circle an isosceles right triangle.

FIG. 365

36. Fig. 365 shows an instrument that may be used to find the center of a metal disk. How is it made and how would it be used?

37. Given the base of an isosceles triangle and the radius of the circumscribed circle, construct the triangle.

38. In Fig. 366, the circle is divided into 5 equal arcs by points A, B, C, D, and E and the chords are drawn as shown. Prove that $\overline{AB} = \overline{BC} = \overline{CD}$ = etc., that $\overline{BE} = \overline{AD} = \overline{CE}$, and that $\angle A = \angle E$.

NOTE. You will learn later how to construct Fig. 366.

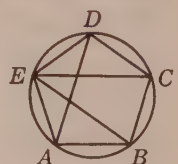


FIG. 366

39. Prove (Fig. 366) that BE and CE trisect $\angle AED$.

A Exercises

1. Prove that the center of the circle circumscribed about an equilateral triangle is also the center of the inscribed circle.

2. Construct the inscribed, the circumscribed, and the three escribed circles (see note, p. 209, Ex. 8) of an equilateral triangle. Prove that the radius of the inscribed circle is one-half the radius of the circumscribed circle and one-third the radius of the escribed circles.

Use problem 12, p. 232, in the next three exercises.

3. Circumscribe an isosceles triangle about a circle, given the base of the triangle. Is the problem always possible?

4. Circumscribe an isosceles triangle about a circle, given the altitude of the triangle. Is the problem always possible?

5. Circumscribe about a given circle a right triangle, given one leg. Is the problem always possible?

Problem 13, p. 239, is to be used in Exs. 6–9. Construct the triangles called for and give full discussion in each case.

Construct a triangle, given

‡**6.** The base, the vertex angle, and the median from the vertex to the base.

‡**7.** The base, the vertex angle, and the altitude.

‡**8.** The base, the vertex angle, and one base angle.

‡**9.** The base, the vertex angle, and one side.

10. In Fig. 367, AC and BD are tangent to $\odot O$ at opposite ends of the diameter AB. CD is an arbitrary tangent intersecting AC and BD at C and D respectively. Prove that $\angle COD$ is a right angle.

Suggestion. If X is the point of contact of tangent CD, draw OX, OC, and OD and prove $\angle 2 + \angle 3 = \frac{1}{2}$ the straight angle.

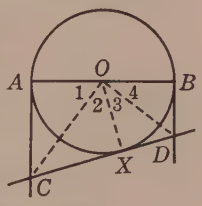


FIG. 367

11. A circle constructed on side AB of $\triangle ABC$ as diameter passes through the feet of the perpendiculars from A and B to the sides BC and AC respectively.

‡ Involves optional work.

12. Circles constructed on two sides of a triangle as diameters will intersect on the third side.

13. A circle constructed on one leg of an isosceles triangle as a diameter passes through the mid-point of the base.

14. If semicircles are constructed on the sides of an equilateral triangle as diameters, they will intersect at the mid-points of the opposite sides (Fig. 368).

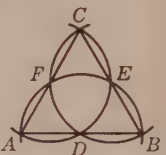


FIG. 368

15. Given a circle (Fig. 369) divided into eight equal parts and the points joined as indicated, prove

a. $AK = KB = BY$.

b. $\angle A = \angle B = \angle C$.

c. $AK = KP$.

d. $OPQRSTUV$ is a regular octagon.

e. $WXYZ$ is a square.

Find the sum of the angles in the points of the star.

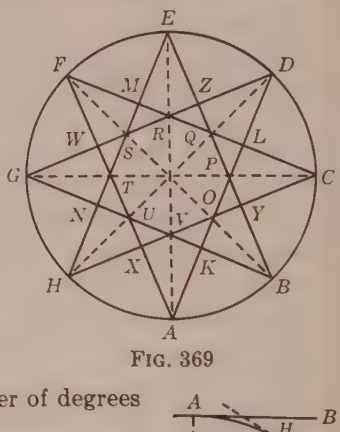


FIG. 369

16. In Fig. 369 find the number of degrees in $\angle XAK$ and $\angle HVB$.

NOTE. Stars similar to that in Fig. 369, often very complicated, are common in cut-glass designs.

17. Fig. 370 shows a simple turnout used on street railroad tracks. The rails $\overset{\frown}{AE}$ and $\overset{\frown}{CF}$ are arcs of circles tangent to rails AB and CD at A and C respectively. Rail AE crosses rail CD at H. Prove that $\angle 1$ made by rail CD and the tangent to AE at H is equal to $\angle 2$. O is the center for $\overset{\frown}{AE}$ and $\overset{\frown}{CF}$.

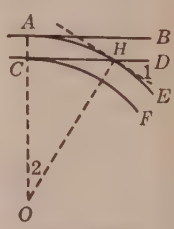


FIG. 370

18. Fig. 371 shows two opposite turnouts from the end of a straight track. The curved rails are tangent to the straight ones at C and D. Prove the angle between the tangents at F is equal to $\angle X + \angle Y$ if X and Y are the centers for the arcs.

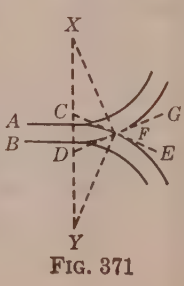


FIG. 371

UNIT V

RATIO AND PROPORTION

256. Uses of ratio. Although you may think that ratio is a term that you will meet only in mathematics, it really is an important factor in the practical affairs of life. Indeed, ratio is involved whenever comparisons of quantity or size are made. You will recall that most maps in your geographies were drawn to scale. That is, a line an inch long represented a certain number of miles. Thus by measuring distance in inches on the map you were able to determine distance in miles on the surface of the earth. Blueprints and architects' plans, which are also drawn to scale, are necessary for exact construction work. In general, ratio is concerned with similarity. This will all be made clear in this unit, in which you will learn that in similar figures the ratios of corresponding parts are equal.

Exercise. Draw a rectangle 3 in. $\times$6 in. How would you draw a second rectangle that has the same shape as the first?

TOPIC 1. THEORY OF RATIO AND PROPORTION

257. Ratio. The **ratio** of two numbers is the relation expressed by dividing one number by the other. The ratio of a to b is written in two forms, $a:b$ or $\frac{a}{b}$. It is read the **ratio of a to b.**

When one says the ratio of 12 to 4 is 3, or $\frac{12}{4}=3$, one means that 12 is 3 times 4; when one says the ratio of a to b is r, or $\frac{a}{b}=r$, one means that a is r times b, or $a=br$.

The quotient r is sometimes called the **value of the ratio.** It is the common practice to use the term **ratio** to mean

either the indicated relation or the quotient r. Two ratios with the same value are said to be **equal.**

258. *Ax. 15. Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

This axiom may be called the fundamental law of ratios. Axiom 5, §51, p. 30, is frequently used for ratios.

259. Proportion. A statement of equality between two ratios is called a **proportion**; that is, if two ratios are equal, the four numbers involved are **in proportion.**

Illustration. The ratio $\frac{2}{4}$ and the ratio $\frac{3}{6}$ are equal. The statement $\frac{2}{4}=\frac{3}{6}$ is a proportion. Make a proportion using the ratio $\frac{1}{3}$, one using $\frac{3}{4}$, one using $\frac{2}{5}$.

260. A proportion may be written in two forms, $a:b=c:d$ or $\frac{a}{b}=\frac{c}{d}$, and is read, "The ratio of a to b equals the ratio of c to d." The four numbers are called **terms** and are numbered in order; a is the first term, b the second, c the third, and d the fourth. The first and fourth terms are called the **extremes** and the second and third are called the **means** of the proportion.

Exercise. Find, in the pages that follow, illustrations of proportions. Tell which are the means and which are the extremes in each case.

261. If two or more ratios are equal, as $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$, then a, b, and c are said to be proportional to x, y, and z.

Fundamental Theorems of Proportion

262. *If four numbers are in proportion, the product of the means equals the product of the extremes.*

Hypothesis: $\frac{a}{b}=\frac{c}{d}$. *Conclusion*: $ad=bc$. Use algebra.

Exercise. Find illustrations of proportions in the pages that follow and apply the above theorem to each.

263. *If the product of two numbers equals the product of two other numbers, either pair of factors may be made the extremes and the other pair the means of a proportion.*

Hypothesis: $ay = bx$. *Conclusion*: $\frac{a}{x} = \frac{b}{y}$.

Suggestion: Divide both sides of the equation by xy.

Exercise. Write as many proportions as you can from each of the equal products:

$2 \times 9 = 3 \times 6$ $\quad 5 \times 6 = 3 \times 10$ $\quad ab = xy$

264. *If three terms of one proportion are equal respectively to three corresponding terms of another proportion, the fourth terms are equal.*

Hypothesis: $\frac{a}{b} = \frac{c}{x}$ and $\frac{a}{b} = \frac{c}{y}$. *Conclusion*: $x = y$.

Suggestion: Apply §262 and use algebra.

Transformations of Proportions

265. *If four numbers are in proportion, the first is to the third as the second is to the fourth;* that is, they are in proportion by **mean alternation.**

Hypothesis: $\frac{a}{b} = \frac{x}{y}$. *Conclusion*: $\frac{a}{x} = \frac{b}{y}$.

Suggestion: First prove $ay = bx$ and then use §263.

266. *If four numbers are in proportion, the fourth is to the second as the third is to the first;* that is, they are in proportion by **extreme alternation.**

Hypothesis: $\frac{a}{b} = \frac{x}{y}$. *Conclusion*: $\frac{y}{b} = \frac{x}{a}$.

267. *If four numbers are in proportion, the second is to the first as the fourth is to the third;* that is, they are in proportion by **inversion.**

Hypothesis: $\frac{a}{b}=\frac{x}{y}$. *Conclusion*: $\frac{b}{a}=\frac{y}{x}$.

268. *If four numbers are in proportion, the first plus the second is to the second as the third plus the fourth is to the fourth;* that is, they are in proportion by **addition.** This is sometimes called **proportion by composition.**

Hypothesis: $\frac{a}{b}=\frac{x}{y}$. *Conclusion*: $\frac{a+b}{b}=\frac{x+y}{y}$.

Suggestion: Adding 1 to each side gives $\frac{a}{b}+1=\frac{x}{y}+1$.

269. *If four numbers are in proportion, the first minus the second is to the second as the third minus the fourth is to the fourth;* that is, they are in proportion by **subtraction.** This is sometimes called **proportion by division.**

Hypothesis: $\frac{a}{b}=\frac{x}{y}$. *Conclusion*: $\frac{a-b}{b}=\frac{x-y}{y}$.

Suggestion: Subtract 1 from each side of the given equation.

C Exercises

1. Write the proportion $\frac{8}{18}=\frac{28}{63}$ (*a*) by mean alternation; (*b*) by extreme alternation; (*c*) by inversion; (*d*) by addition; (*e*) by subtraction. Verify each by finding the values of the resulting ratios.

2. Work Ex. 1 using each of the following:

$\frac{15}{20}=\frac{9}{12}$; $\frac{18}{32}=\frac{27}{48}$; $\frac{20}{35}=\frac{16}{28}$; $\frac{ay}{by}=\frac{ax}{bx}$; $\frac{ay}{a^2}=\frac{by}{ab}$.

3. Take any proportion, then write it:

a. First by mean alternation and that result by addition.

b. First by extreme alternation and that result by subtraction.

c. First by inversion and that result by addition.

d. First by addition and that result by extreme alternation.

e. First by subtraction and that result by inversion.

4. If $\frac{a}{b}=\frac{x}{y}$, show how each of the following is obtained:

a. $\frac{a+b}{a}=\frac{x+y}{x}$. *c*. $\frac{a+b}{x+y}=\frac{a}{x}$. *e*. $\frac{a-b}{x-y}=\frac{b}{y}$.

b. $\frac{a}{a-b}=\frac{x}{x-y}$. *d*. $\frac{a-b}{x-y}=\frac{a}{x}$. *f*. $\frac{a+b}{a-b}=\frac{x+y}{x-y}$.

Suggestion: For *f*, write the given proportion first by addition and then by subtraction. Divide the first result by the second.

270. Measurement of segments. If three equal segments a are added to make the segment b, it is said that the segment b is three times the segment a, or that the ratio of segment b to segment a is 3. You may regard segment a as a unit of measure. Then you call 3 the measure number of b in terms of the unit a.

271. To measure a segment, you find how many times it contains some other segment which is taken as a unit.

The number of times that the segment to be measured contains the unit is called the **measure number** or simply the **measure** or the **length** of the segment.

272. Any arbitrary segment may be taken as a unit of measure. In practical work in the United States the inch, foot, yard, rod, mile, centimeter, and meter are used. In printing, the em and the pica are used.

273. In measuring segments three possibilities arise:

a. When a segment is to be measured with a given unit, it may happen that the unit will be contained in the segment an integral number of times with no remainder. The measure of the yard when one foot is used as a unit is 3. The measure number is an integer.

b. In other cases there may be a remainder. In such cases it is sometimes possible to divide the unit into equal parts so that one of them will measure the segment without a remainder. The measure number is a fraction.

Illustration 1. The segment $3\frac{1}{2}$ in. will contain the unit one-half inch 7 times without a remainder. The measure number is the integer 7.

Illustration 2. If a rod is measured with a yard as a unit, the measure number is a fraction, $\frac{11}{2}$ yds., or $5\frac{1}{2}$ yds.

c. It may happen that the unit of measure cannot be divided so as to find a new unit that will be contained exactly in the given segment. You still say that there is a measure number for the given segment, but in this case it is a new kind of number called an **irrational number.**

Illustration. You may try to measure the diagonal of a square using one side of the square as a unit. There is a remainder, but no fraction of the side can be found that will measure the diagonal exactly. The measure of the diagonal is the irrational number $\sqrt{2}$.

274. Ratios of segments. By the **ratio of two segments** is meant the ratio of their measure numbers when expressed in the same unit.

Exercise. What is the ratio of segments (*a*) $2\frac{1}{2}$ in. and 3 in. using one-half inch as a unit? (*b*) $1\frac{1}{4}$ in. and $3\frac{1}{2}$ in. using one-quarter inch as a unit?

275. If a common measure of two segments can be found, the segments are said to be **commensurable.** The segments mentioned in each part of the last exercise are commensurable. The ratio of two commensurable segments is an integer or a common fraction.

276. If a common measure of two segments cannot be found, they are said to be **incommensurable.** Their ratio is an irrational number.

277. A **rational number** is a number that can be changed into an integer or a common fraction. An **irrational number** cannot be changed into an integer or a common fraction. π and inexact roots like $\sqrt{2}$ are irrational numbers.

278. You will later meet several cases of incommensurable segments. The following are illustrations:

a. The side of a square and its diagonal. Their ratio always contains $\sqrt{2}$.

b. The side of an equilateral triangle and its altitude. Their ratio always contains $\sqrt{3}$.

c. The circumference and the diameter of a circle. Their ratio always contains π.

NOTE. The nature of decimals that correspond to rational and to irrational numbers is interesting and should be noted. Rational numbers reduce to decimals that either terminate or repeat, for example:

$\frac{1}{16} = .0625$ $\frac{1}{6} = .1666$ or $.1\dot{6}$. $\frac{1}{3} = .333$ or $.\dot{3}$.

Irrational numbers give decimals that neither terminate nor repeat. An inexact root like $\sqrt{2}$ or $\sqrt{3}$ is irrational. The decimals given in the table for these numbers (p. 430) are approximations only. There are other kinds of irrational numbers, but they all give decimals that neither terminate nor repeat.

279. In actual practice an exact measurement can never be obtained. If you try to measure the segment, the end will fall between two marks on the scale. Either of these gives an approximation to the length of the segment, one a little too small and one a little too large. It is the usual practice to use the nearest one as the approximate length of the segment. Often extremely close approximations are necessary. The degree of accuracy depends upon the definiteness of the end of the segment to be measured and upon the fineness of the scale. Sometimes units as small as .01 or .001 of an inch are used.

TOPIC 2. EQUAL RATIOS—METHOD I

THEOREM 61

280. If a line is parallel to the base of a triangle, the ratio of the segments on one side equals the ratio of the corresponding segments on the other side.

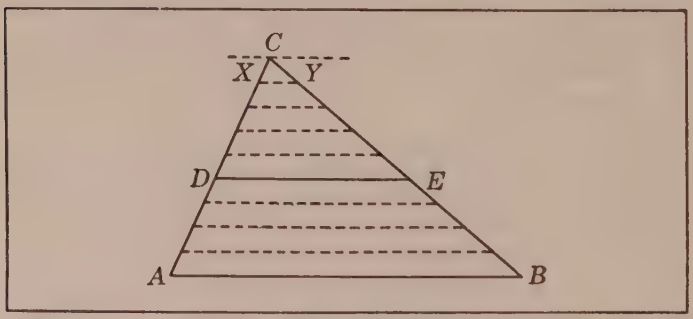


FIG. 372

Hypothesis: $\triangle ABC$ is any $\triangle$. $DE \parallel AB$.

Conclusion: $\frac{CD}{DA} = \frac{CE}{EB}$.

Case A. When CD and DA are commensurable.

Construction and general plan:

1. The ratio of $\frac{CD}{DA}$ and of $\frac{CE}{EB}$ are to be obtained from the measure numbers of the segments.
2. For CD and DA, assume CX as the common unit of measure and apply it to CD and DA.
3. For CE and EB, draw lines through C and the points of division on $CA \parallel AB$ and let them intersect CB.

Outline of proof. Give the reasons.

1. *a.* CX is the common unit of measure for CD and DA.
   b. Let CX be contained in CD m times.

c. $\therefore$ m is the measure number of CD.

d. Let CX be contained in DA n times.

e. $\therefore$ n is the measure number of DA.

f. $\therefore$ the ratio of $\frac{CD}{DA} = \frac{m}{n}$ (see §274, p. 258).

2. *a.* The divisions on CB made by the parallels are equal (see Th. 28, p. 142).

b. CY is the common unit of measure for CE and EB.

c. CY is contained in CE m times.

d. $\therefore$ m is the measure number of CE.

e. CY is contained in EB n times.

f. $\therefore$ n is the measure number of EB.

g. $\therefore$ the ratio of $\frac{CE}{EB} = \frac{m}{n}$.

3. $\therefore$ the ratio of $\frac{CD}{DA}$ = the ratio of $\frac{CE}{EB}$ (see Ax. 5, p. 30).

NOTE. The fact that the number of divisions on CE and EB (2*c* and 2*e* above) are equal respectively to the number of divisions on CD and DA, may be inferred directly from the figure.

Case B. When CD and DA are incommensurable. Since it is impossible in this case to express the lengths of CD and DA in integral or fractional terms of the same unit, the argument given for Case A cannot be used. Case B will be assumed without proof. The proof is possible, but too difficult for this course.

281. COR. I. **If a line is parallel to the base of a triangle, the ratio of one side to either of its segments equals the ratio of the other side to its corresponding segment.**

Suggestion. Write the proportion proved in Th. 61 by (*a*) addition and (*b*) inversion and addition, and note the results on the figure.

Ex. 1. What ratios are equal in Fig. 373 by Th. 61, p. 260? Write this proportion by (*a*) mean alternation, (*b*) extreme alternation, (*c*) inversion. Note the results on the figure.

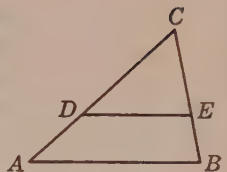


FIG. 373

Ex. 2. Work Ex. 1 for the ratios proved equal in Cor. I, p. 261.

282. COR. II. **If three parallels cut two transversals, the segments on one transversal have the same ratio as the corresponding segments on the other transversal.**

Analysis: To prove $\frac{AB}{BC}=\frac{DE}{EF}$, prove each ratio equal to $\frac{DX}{XC}$ (Fig. 374).

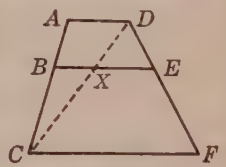


FIG. 374

Ex. 1. Work Ex. 1 above for the proportion proved in Cor. II.

Ex. 2. Write the proportion proved in Cor. II by (*a*) addition, (*b*) inversion and addition. Note the results on the figure.

C Exercises

In the following exercises the letters refer to Fig. 373:

1. $CD=20$, $DA=15$, $CE=28$, find EB.
2. $CD=27$, $DA=8$, $EB=12$, find CE.
3. $DA=48$, $CE=45$, $EB=60$, find CD.
4. $CD=16$, $DA=14$, $CB=45$, find CE and EB.
5. $CA=24$, $CD=15$, $CB=36$, find CE and EB.
6. $CA=49$, $DA=35$, $CB=77$, find CE and EB.
7. $CD=56$, $DA=40$, $CE=35$, find CB.
8. $DA=2.3$, $CE=5.7$, $EB=9$, find CA.
9. $CD=5\frac{1}{3}$, $CE=4\frac{4}{5}$, $EB=2\frac{2}{5}$, find DA.
10. $CA=8\frac{1}{4}$, $DA=2\frac{3}{4}$, $CB=5$, find CE and EB.
11. $CA=8\frac{2}{3}$, $CD=3\frac{1}{3}$, $EB=6\frac{1}{2}$, find CB.
12. $DA=3\frac{3}{5}$, $CE=3\frac{3}{4}$, $EB=6\frac{3}{4}$, find CD.

Problem 14

283. To divide a given segment into two parts that shall have the same ratio as two given segments.

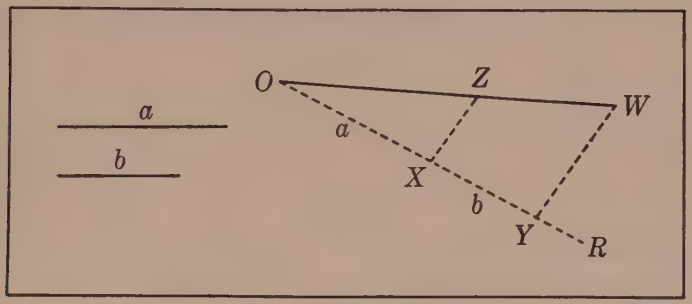


Fig. 375

Given OW any segment, a and b two given segments.

To divide OW into parts that shall have the same ratio as a and b.

Directions:

1. Draw the ray OR making any angle with OW.
2. On OR lay off $OX = a$ and $XY = b$.
3. Join Y and W and from X draw XZ parallel to YW meeting OW at Z.
4. OZ and ZW are the required parts.

Proof: Prove $\frac{OZ}{ZW} = \frac{a}{b}$ by Th. 61, p. 260.

Ex. 1. By algebra divide 96 into parts that shall have the ratio 7:9.

Ex. 2. By Prob. 7, page 143, find two-ninths of a given segment.

Ex. 3. Divide a segment 5 cm. long into parts that shall have the ratio 1:2. Use Prob. 14.

Ex. 4. By Prob. 14 find two-thirds of a segment 5 cm. long.

Exercises B

1. Divide a given segment into parts proportional to three given segments (Fig. 376).

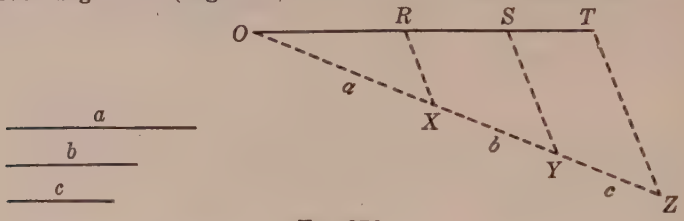


FIG. 376

Give the directions.

Analysis for the proof: Prove $\frac{OR}{a}=\frac{RS}{b}=\frac{ST}{c}$.

I. To prove $\frac{OR}{a}=\frac{RS}{b}=\frac{ST}{c}$, prove $\frac{OR}{a}=\frac{RS}{b}$ and $\frac{RS}{b}=\frac{ST}{c}$.

II. To prove $\frac{OR}{a}=\frac{RS}{b}$, use Th. 61, p. 260, and mean alternation.

III. To prove $\frac{RS}{b}=\frac{ST}{c}$, use §282, p. 262, and mean alternation.

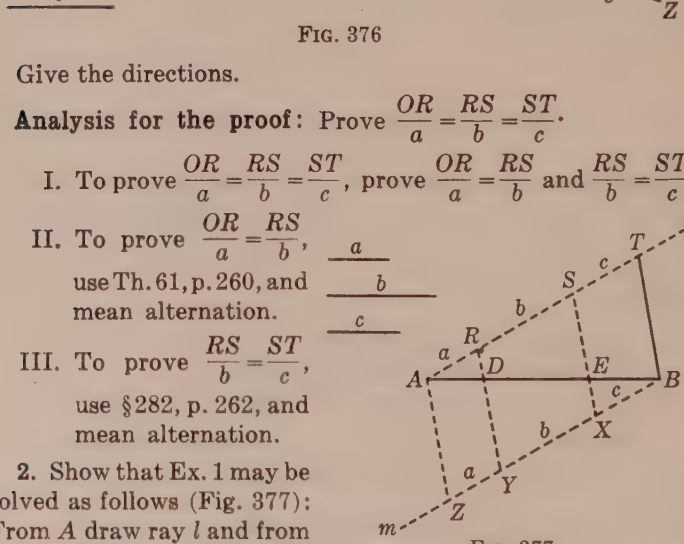


FIG. 377

2. Show that Ex. 1 may be solved as follows (Fig. 377): From A draw ray l and from B draw $m \parallel l$. Lay off the given segments as shown in the figure: $AR = ZY = a$, $RS = YX = b$, and $ST = XB = c$. Join the points of division.

3. In Fig. 378, $BE \parallel CF \parallel DG$. Prove $\frac{AB}{AE}=\frac{AC}{AF}=\frac{AD}{AG}$.

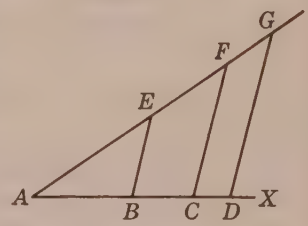


FIG. 378

Problem 15

284. To find the fourth segment in a proportion when the other three segments are given in order.

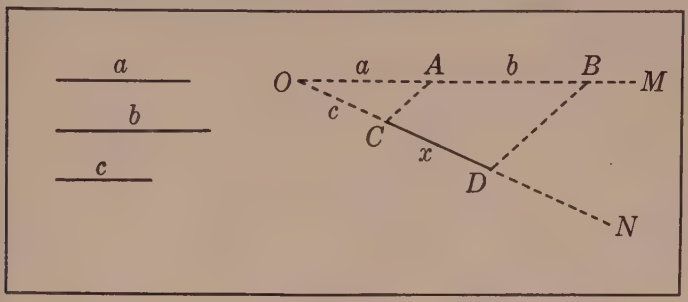


Fig. 379

Given three segments a, b, and c.

To construct segment x so that $\frac{a}{b}=\frac{c}{x}$.

Analysis: To construct x so that $\frac{a}{b}=\frac{c}{x}$, construct a line parallel to the base of a triangle so that a and b are segments on one side taken in order from the vertex and c and x are the segments on the other side.

Directions:

1. Draw $\angle MON$ any angle.
2. From O on OM lay off $OA=a$.
3. From A on OM lay off $AB=b$.
4. From O on ON lay off $OC=c$.
5. Draw AC and from B draw $BD \parallel AC$.
6. CD is the required segment.

Proof: Prove $\frac{a}{b}=\frac{c}{x}$. Show that Th. 61, p. 260, applies.

THEOREM 62*

285. If a line divides the sides of a triangle so that one side is to one segment as a second side is to its corresponding segment, the line is parallel to the third side of the triangle.

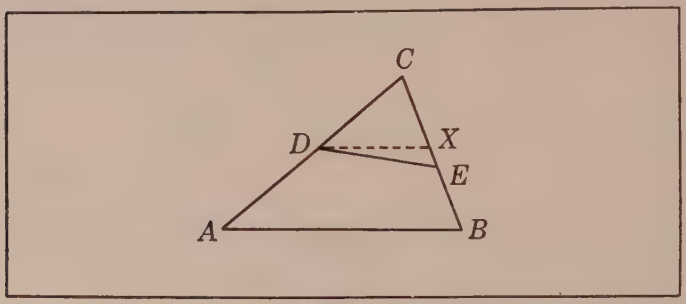


FIG. 380

Hypothesis: $\triangle ABC$ is any triangle with DE drawn so that $\frac{CA}{CD}=\frac{CB}{CE}$.

Conclusion: $DE \parallel AB$.

Analysis and construction:

1. To prove $DE \parallel AB$, prove that DE coincides with a line $\parallel AB$.
2. $\therefore$ construct DX from $D \parallel AB$ and prove that DE coincides with DX.
3. To prove that DE coincides with DX, prove that E falls on X.
4. To prove that E falls on X, prove that $CE=CX$.
5. To prove that $CE=CX$, show that $\frac{CA}{CD}=\frac{CB}{CX}$ and that $\frac{CA}{CD}=\frac{CB}{CE}$.

Proof:

	STATEMENTS	REASONS
1.	$\frac{CA}{CD}=\frac{CB}{CE}$.	1. Given.
2.	$\frac{CA}{CD}=\frac{CB}{CX}$.	2. §281, p. 261.
3.	$CE=CX$.	3. §264, p. 255.
4.	E falls on X.	4. See Post. 45 below.
5.	DE coincides with DX.	5. Post. 2, p. 19.
6.	$DE \parallel AB$.	6. See analysis, 1.

286. *Post. 45. If two equal segments begin at the same point and extend along the same line in the same direction, they end at the same point.*

C Exercises

1. Name the pairs of segments having equal ratios when a line parallel to the bases of a trapezoid cuts the non-parallel sides and the diagonals.

2. Find by arithmetic the fourth term of a proportion in which the other three are the numbers given below taken in order:

a. 21, 5, and 14
b. 4.2, 2.5, and 3.7
c. $6\frac{1}{2}$, 26, and 35
d. $6\frac{1}{3}$, $8\frac{2}{3}$, and 5

3. Work part *b* in Ex. 2 by geometry as in Prob. 15, p. 265, using 4.2 cm., 2.5 cm., and 3.7 cm., and measure your result. Compare with your answer obtained by arithmetic.

4. In Fig. 381, $ED \parallel CB$ and $CF \parallel EB$. Prove (*a*) $\frac{AD}{DB}=\frac{AB}{BF}$, (*b*) $\frac{AD}{AB}=\frac{AB}{AF}$, and (*c*) $\frac{AB}{DB}=\frac{AF}{BF}$.

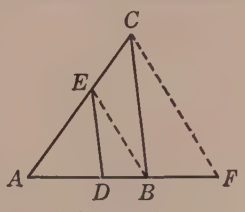


FIG. 381

Analysis: To prove the two ratios equal, prove them each equal to a third ratio. (Name the third ratio in each case.)

B Exercises

1. Using any three given segments a, b, and c, construct a fourth segment x so that (*a*) $\frac{a}{c}=\frac{b}{x}$; (*b*) $\frac{a}{b}=\frac{c}{x}$; (*c*) $\frac{c}{a}=\frac{b}{x}$; (*d*) $\frac{b}{c}=\frac{a}{x}$; (*e*) $\frac{c}{b}=\frac{a}{x}$; (*f*) $\frac{b}{a}=\frac{c}{x}$.

2. Do any of the figures called for in Ex. 1 give the same length for x? Why?

3. Find x by arithmetic in each of the proportions if $a=3\frac{1}{3}$, $b=2\frac{1}{2}$, $c=2\frac{2}{5}$.

4. If a, b, and c are three given segments, find a fourth segment x so that (*a*) $x=\frac{bc}{a}$; (*b*) $x=\frac{2bc}{a}$; (*c*) $x=\frac{a^2}{b}$.

5. In Fig. 382, O is any point within $\triangle ABC$. $A'B' \parallel AB$ from A', an arbitrary point in AO, and intersects OB at B'. $B'C' \parallel BC$ from B' and intersects OC at C'. A' and C' are joined. Prove that $A'C' \parallel AC$.

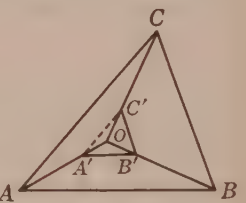


FIG. 382

6. Prove Ex. 5 when point O is outside the triangle.

7. Prove by proportion that a line parallel to one side of a triangle and bisecting a second side bisects the third.

8. Prove by proportion that a line bisecting one of the nonparallel sides of a trapezoid and parallel to the bases bisects the other.

287. Internal and external division. (*Optional.*) If point C is between A and B on segment AB, then AB is said to be divided internally at C. $AC+CB=AB$ (Fig. 383). If C' is on the segment AB prolonged, AB is said to be divided externally at C'. $AC'-BC'=AB$ (Fig. 384). In one case AC and CB and in the other case AC' and BC' are said to be segments of AB.

FIG. 383

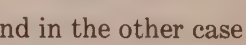


FIG. 384

THEOREM 63 (*Optional*)

288. The bisector of an angle of a triangle divides the opposite side into segments that have the same ratio as the other two sides of the triangle.

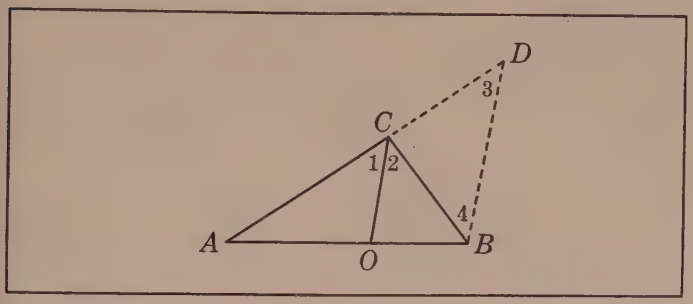


FIG. 385

Hypothesis: $\triangle ABC$ is any triangle; CO bisects $\angle ACB$ and divides the base AB into segments AO and OB.

Conclusion: $\frac{AO}{OB}=\frac{AC}{CB}$.

Analysis and construction:

1. To prove $\frac{AO}{OB}=\frac{AC}{CB}$, use a line parallel to side OC of $\triangle AOC$.
2. $\therefore$ from B draw $BD \parallel OC$ to meet AC extended at D and prove $\frac{AO}{OB}=\frac{AC}{CD}$ and $CB=CD$.
3. To prove $\frac{AO}{OB}=\frac{AC}{CD}$, use Th. 61, p. 260.
4. " " $CB=CD$, prove $\angle 3=\angle 4$.
5. " " $\angle 3=\angle 4$, " $\angle 3=\angle 1$ and $\angle 4=\angle 2$.

Exercise. (*Optional.*) Compare Ths. 63 and 64 and show how the definitions in §287 apply.

THEOREM 64 (*Optional*)

289. The bisector of an exterior angle of a triangle not isosceles divides the opposite side externally into segments that have the same ratio as the other two sides of the triangle.

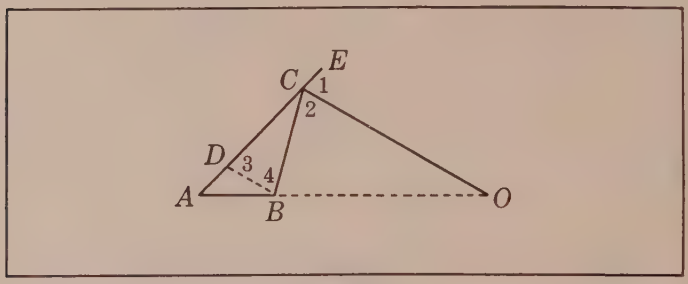


FIG. 386

Hypothesis: $\triangle ABC$ is any triangle not isosceles; CO bisects the exterior $\angle BCE$ and divides the base AB externally into segments OA and OB.

Conclusion: $\frac{OA}{OB}=\frac{CA}{CB}$.

Analysis and construction:

1. To prove $\frac{OA}{OB}=\frac{CA}{CB}$, use a line parallel to line CO.

2. $\therefore$ from B draw $BD \parallel OC$ to meet AC at D and prove $\frac{OA}{OB}=\frac{CA}{CD}$ and $CD=CB$.

3. To prove $CD=CB$, prove $\angle 3=\angle 4$.

Ex. 1. Prove Th. 63, p. 269, by extending AC so that $CD=CB$ (Fig. 385). Join D and B and prove $DB \parallel CO$.

Ex. 2. (*Optional.*) Prove Th. 64 by making $CD=CB$ (Fig. 386). Join D and B and prove $DB \parallel CO$.

TOPIC 3. EQUAL RATIOS—METHOD II

290. Similar figures. In popular language, similar figures are figures that have the same shape. Look through the book and find illustrations of such figures, as on pages 272 and 342.

291. Two polygons are said to be **similar** if

(1) The angles of one are equal to the corresponding angles of the other and

(2) The corresponding sides have equal ratios.

If it is necessary to prove two polygons similar, it is necessary to prove both of the facts given above. Conversely, if two polygons are given similar, both of the facts given above are true. $\sim$ means similar.

Ex. 1. If it is required to prove $\triangle ABC \sim \triangle A'B'C'$ (Fig. 387), what must be proved?

Ex. 2. If, in Fig. 387, $\triangle ABC$ is given similar to $\triangle A'B'C'$, what facts do you know about the triangles?

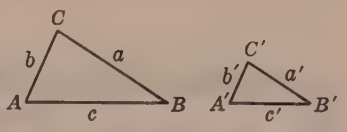


FIG. 387

Ex. 3. Turn to figures farther on in the text and find illustrations of polygons that appear to be similar. Answer the questions in Exs. 1 and 2 for these polygons.

NOTE. In selecting the similar polygons called for in Ex. 3, select some with more than three sides. Refer to the letters of the figure.

If the triangles in Fig. 387 are similar:

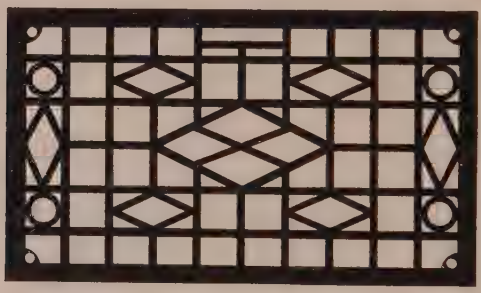
Ex. 4. Find b' and c', if $a=8$, $b=12$, $c=18$, and $a'=10$.

Ex. 5. Find a' and c', if $a=3\frac{1}{2}$, $b=4$, $c=5\frac{1}{3}$, and $b'=64$.

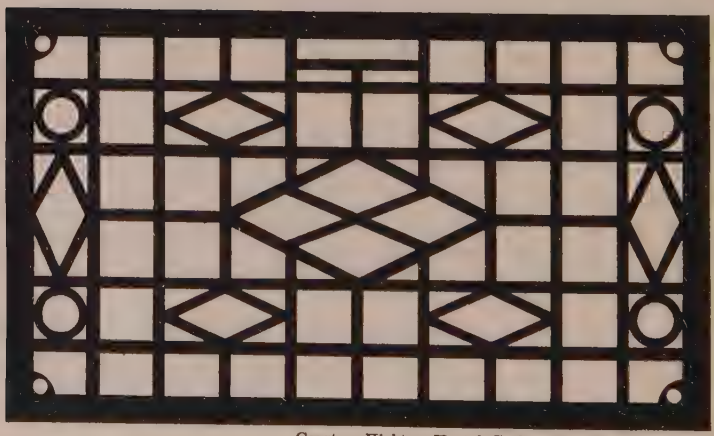
Ex. 6. Find b and c, if $a=7\frac{1}{2}$, $a'=12$, $b'=16$, and $c'=20$.

292. The ratio of any two corresponding sides of two similar polygons is called the **ratio of similitude** of the polygons.

Exercise. Select in the two figures below illustrations of similar polygons and of polygons of the same number of sides that are not similar.



No. 1



Courtesy Highton, Hart & Cooley, Grilles and Registers

No. 2

FIG. 388. **Ornamental grille.** No. 1 is a drawing of an ornamental grille. No. 2 is an enlargement with the ratio of similitude 2 : 3.

THEOREM 65*

293. Two mutually equiangular triangles are similar.

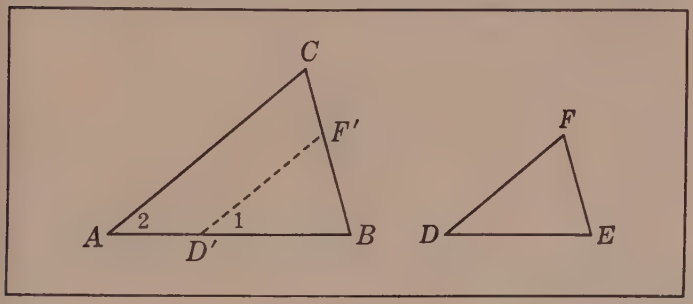


FIG. 389

Hypothesis: $\triangle ABC$ and $\triangle DEF$ are any two triangles with $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

Conclusion: $\triangle ABC \sim \triangle DEF$.

Analysis and construction. Give the proof.

1. *a.* To prove $\triangle ABC \sim \triangle DEF$, prove $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

   b. $\therefore$ prove $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{BC}{EF} = \frac{CA}{FD}$.

2. *a.* To prove $\frac{AB}{DE} = \frac{BC}{EF}$, place $\triangle DEF$ upon $\triangle ABC$ with point E on point B, ED along BA, EF along BC, and prove $\frac{AB}{D'B} = \frac{BC}{BF'}$.

   b. To prove $\frac{AB}{D'B} = \frac{BC}{BF'}$, prove $D'F' \parallel AC$.

   c. $\therefore$ prove $\angle 1 = \angle 2$.

3. Give the analysis for $\frac{BC}{EF} = \frac{CA}{FD}$.

Ex. 1. When are polygons mutually equiangular?

Ex. 2. Prove that all equilateral triangles are similar.

Ex. 3. Are all isosceles triangles similar? Why? When are two isosceles triangles similar? Why?

Ex. 4. Is a square similar to a rectangle? Why?

Ex. 5. Construct two rectangles that are similar.

294. COR. I. **If two triangles have two angles of one equal to two angles of the other, the triangles are similar.**

295. COR. II. **If two triangles have two angles of one equal to two angles of the other, the corresponding sides have equal ratios.**

296. Methods for proving ratios and products equal.

When **two ratios are to be proved equal:**

A. Our fundamental methods for proving ratios equal are:
   1. By parallels and transversals (§§280–282, pp. 260–262).
   2. By similar triangles (explained below).

B. Before either of these methods can be applied, it is often necessary to find a third ratio to which each of the given ratios can be proved equal.

When **two products are to be proved equal,** obtain a pair of equal ratios from the equal products by §263, p. 255, and then prove the ratios equal. The products will be equal by §262, p. 254.

297. To prove ratios equal by similar triangles, note:

a. **The proper triangles must be carefully selected.** In Fig. 390, let a and a', b and b', c and c' be corresponding sides. Then since corresponding sides have equal ratios (§291, p. 271) you have $\frac{a}{a'} = \frac{b}{b'}$. If, as in Fig. 391, you are

to prove $\frac{OA}{OB}=\frac{OX}{OY}$, you may choose the triangles and draw construction lines so that OA and OX shall be sides of

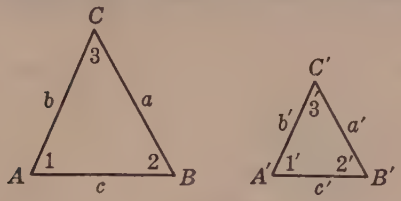


FIG. 390

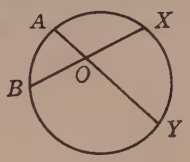


FIG. 391

one triangle and OB and OY sides of the other; that is, so that the numerators shall be sides of one triangle and the denominators sides of the other. If convenient triangles are not obtained, write the proportion by mean alternation and try again. Sometimes, however, triangles cannot be obtained without proving each ratio equal to a third.

b. **The corresponding sides of similar triangles must be carefully selected.** In Fig. 390, the triangles are so placed that corresponding sides may be selected by inspection. Corresponding sides of congruent triangles are usually selected in this way. Similar triangles, however, are often not so conveniently placed. In this case it is necessary to notice that corresponding sides are always opposite equal angles. Read the ratios thus:

$$\frac{a\ (\text{opposite}\ \angle 1)}{a'(\text{opposite}\ \angle 1')}=\text{etc.}$$

It is convenient to number equal angles $\angle 1$ and $\angle 1'$, etc.

c. **Time is saved by using Cor. II to Th. 65** rather than the theorem itself (see §295, p. 274). You need not mention in the proof the fact that the triangles are similar.

NOTE. You may say that corresponding sides are proportional. In this case you may use either $\frac{a}{a'}=\frac{b}{b'}$ or $\frac{a}{b}=\frac{a'}{b'}$.

THEOREM 66*

298. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

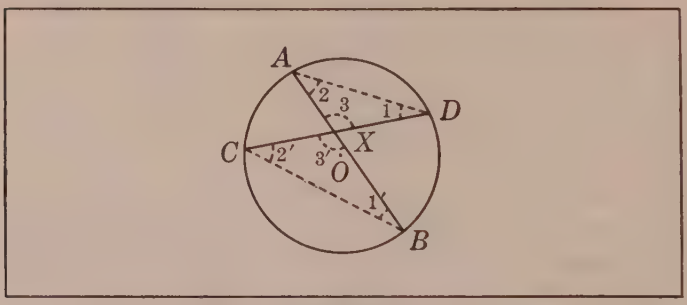


FIG. 392

Hypothesis: Circle O is any circle with the chords AB and CD intersecting at X.

Conclusion: $AX \cdot XB = CX \cdot XD$.

Analysis and construction:

1. To prove $AX \cdot XB = CX \cdot XD$, prove $\frac{AX}{CX} = \frac{XD}{XB}$.
2. " " $\frac{AX}{CX} = \frac{XD}{XB}$, draw AD and CB and use $\triangle AXD$ and $\triangle CXB$.
3. Prove $\angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$.

Outline of proof. Give all reasons in detail.

1. $\angle 1 = \angle 1'$.
2. $\angle 2 = \angle 2'$.
3. $\therefore \frac{AX \text{ (opposite } \angle 1)}{CX \text{ (opposite } \angle 1')} = \frac{XD \text{ (opposite } \angle 2)}{XB \text{ (opposite } \angle 2')}$.
4. $\therefore AX \cdot XB = CX \cdot XD$.

THEOREM 67

299. If two secants intersect without a circle, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

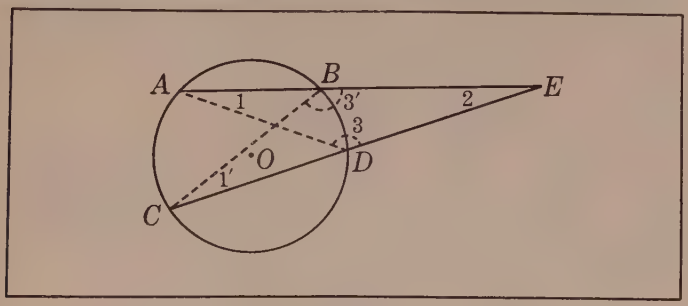


FIG. 393

Hypothesis: Circle O is any circle with the two secants EA and EC intersecting without the circle at E.

Conclusion: $AE \cdot BE = CE \cdot DE$.

Analysis and construction:

1. To prove $AE \cdot BE = CE \cdot DE$, prove $\frac{AE}{CE} = \frac{DE}{BE}$.
2. " " $\frac{AE}{CE} = \frac{DE}{BE}$, draw AD and BC and use $\triangle AED$ and $\triangle CEB$.
3. Prove $\angle 2 = \angle 2$, $\angle 1 = \angle 1'$, and $\angle 3 = \angle 3'$.

Outline of proof. Give all reasons in detail.

1. $\angle 2 = \angle 2$.
2. $\angle 1 = \angle 1'$.
3. $\angle 3 = \angle 3'$.
4. $\therefore \frac{AE \text{ (opposite } \angle 3)}{CE \text{ (opposite } \angle 3')} = \frac{DE \text{ (opposite } \angle 1)}{BE \text{ (opposite } \angle 1')}$.
5. $\therefore AE \cdot BE = CE \cdot DE$.

300. To obtain the triangles. In proving two products equal, the proportion used may be obtained in more than one form (see §263, p. 255). Thus, in Th. 66, p. 276, you are to prove $AX \cdot XB = CX \cdot XD$. Therefore, you may write either $\frac{AX}{CX} = \frac{XD}{XB}$ or $\frac{AX}{XD} = \frac{CX}{XB}$. In Th. 67, p. 277, you are to prove $AE \cdot BE = CE \cdot DE$. Therefore, you may write either $\frac{AE}{CE} = \frac{DE}{BE}$ or $\frac{AE}{DE} = \frac{CE}{BE}$.

The directions given in §297, p. 274, do not give the same triangles for the two forms of the proportion. Choose that form which seems to give the easiest proof. Experiment with Th. 66, p. 276, and Th. 67, p. 277. Does it make any difference which form you use?

301. To obtain the equal angles. Occasionally the equal angles in the two triangles may be obtained from the hypothesis with little or no difficulty. But very often this is not true. Sometimes one can imagine the smaller triangle placed upon the larger so as to obtain a line parallel to one side of the larger. Corresponding angles are then clearly seen. Since the analysis is the proof done backward, one can obtain the equal angles from the ratios to be proved equal. Thus in Fig. 394, p. 279, if you are to prove $\frac{CA}{CD} = \frac{CD}{CB}$, note you are to use $\triangle CDA$ and $\triangle CDB$. CA and CD are to be corresponding sides of these triangles; therefore, $\angle 3$ and $\angle 3'$, which are opposite these sides respectively, are to be corresponding angles.

Exercise. Show how the analyses for Th. 66, p. 276, and Th. 67, p. 277, were worked out.

302. Mean proportional. If the product of two segments equals the square of a third segment, the last segment is a **mean proportional** between the other two.

THEOREM 68

303. If a secant and a tangent meet without a circle, the tangent is a mean proportional between the whole secant and its external segment.

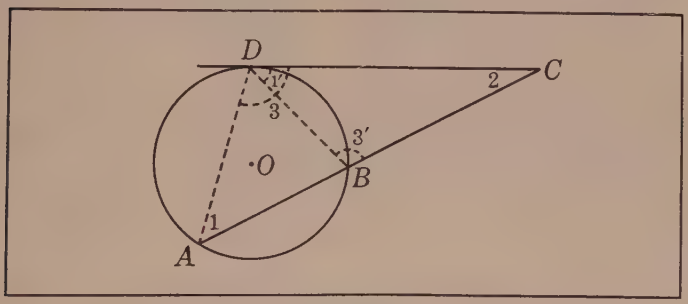


FIG. 394

Hypothesis: Circle O is any circle with secant CA and tangent CD meeting without the circle at C.

Conclusion: CD is a mean proportional between CA and CB.

Analysis and construction:

1. To prove CD a mean proportional between CA and CB, prove $\overline{CD}^2 = CA \cdot CB$.
2. To prove $\overline{CD}^2 = CA \cdot CB$, prove $\frac{CA}{CD} = \frac{CD}{CB}$.
3. " " $\frac{CA}{CD} = \frac{CD}{CB}$, draw AD and BD and use $\triangle CDA$ and $\triangle CDB$.
4. Prove $\angle 2 = \angle 2$, $\angle 1 = \angle 1'$, and $\angle 3 = \angle 3'$.

NOTE. The arrangement and details of the proof are similar to those for Th. 66, p. 276, and Th. 67, p. 277, except that there is an extra step at the end. Why? When proving $\angle 1 = \angle 1'$, use Th. 56, p. 224, Th. 58, p. 228, and Post. 43, p. 229.

B Exercises

1. If you were to prove the following ratios equal, what would be the first step in the analysis? No figure is needed. See §297, *a*, p. 274.

a. $\frac{MB}{NF}=\frac{BC}{CF}$. *b.* $\frac{BE}{BH}=\frac{BM}{BA}$. *c.* $\frac{BE}{BC}=\frac{EM}{BM}$. *d.* $\frac{BC}{BH}=\frac{CM}{AB}$.

2. In Fig. 395, O is any point in segment AB. Any line is drawn through point O not perpendicular to AB. From A and B perpendiculars to XY are drawn meeting XY at Y and X. Prove that $\frac{AY}{BX}=\frac{AO}{BO}$.

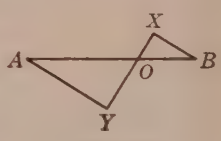


FIG. 395

3. $ABCD$ is a parallelogram with its diagonal AC. BX is a line drawn through B intersecting AC at Y and AD at X. Prove that $\frac{BY}{XY}=\frac{BC}{AX}$.

4. The diagonals of a trapezoid divide each other into segments that have the same ratio.

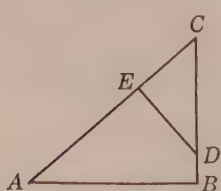


FIG. 396

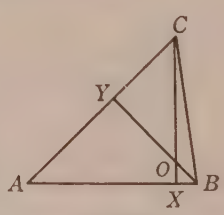


FIG. 397

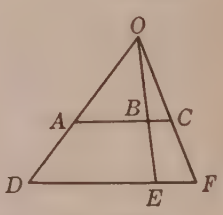


FIG. 398

5. In Fig. 396, ABC is a rt. $\triangle$ with $\angle B$ a rt. $\angle$. DE is drawn $\perp AC$ from any point in BC. Prove $\frac{BC}{EC}=\frac{AC}{DC}$.

6. Prove Ex. 5 when point D is on CB extended.

7. Prove Ex. 5 when point D is on BC extended.

8. In Fig. 397, $CX \perp AB$ and $BY \perp AC$, prove $\frac{AC}{AB} = \frac{CX}{BY}$.

9. Write one or more pairs of equal ratios from the following equal products. Use §263, p. 255.

a. $AX \cdot XZ = XY \cdot XB$.
b. $AB \cdot XZ = BC \cdot XY$.
c. $HK \cdot QK = AH \cdot KN$.
d. $RS \cdot NQ = MQ \cdot ST$.
e. $PQ \cdot ST = RS \cdot QR$.
f. $PX \cdot YQ = XQ \cdot YP$.

10. If the ratios obtained in answer to Ex. 9 were to be proved equal, what triangles would you use in each case?

11. In Fig. 398, if $AC \parallel DF$, prove $OD \cdot OC = OF \cdot OA$.

12. Prove the converse to Ex. 11.

13. If (Fig. 397), $CX \perp AB$ and $BY \perp AC$, prove $BO \cdot BY = BX \cdot BA$.

14. In Fig. 398, AC and DF are parallel and are cut by rays from point O as shown. Prove $\frac{AB}{DE} = \frac{BC}{EF}$.

Suggestion. Prove each ratio equal to a third.

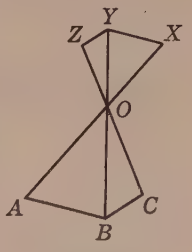


FIG. 399

15. In Fig. 399, $AB \parallel XY$ and $BC \parallel YZ$. Prove $\frac{AB}{XY} = \frac{BC}{YZ}$.

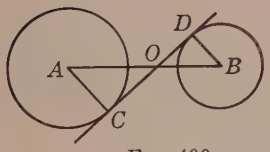


FIG. 400

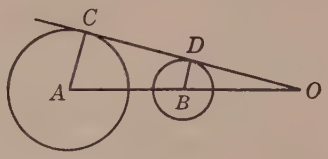


FIG. 401

16. The common tangent to two circles divides the segment joining the centers into segments that have the same ratio as the radii. Prove two cases. See Figs. 400 and 401. See also §287, p. 268.

17. Draw any triangle and its three altitudes. Find all possible pairs of similar triangles and read the ratios of corresponding sides.

18. In Fig. 402, ABC is any triangle inscribed in the circle. CQ bisects $\angle C$. Prove that $CA \cdot CB = CP \cdot CQ$.

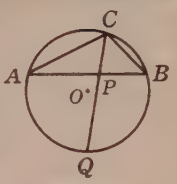


FIG. 402

19. Extend the sides of any triangle through a vertex and draw a line parallel to the base cutting these extended sides. Prove the two triangles similar and write the ratios of the corresponding sides.

20. In Fig. 403, P is the mid-point of the arc CD. PA and PB are arbitrary chords intersecting chord CD at X and Y respectively. Prove that $PY \cdot PB = PX \cdot PA$.

FIG. 403

A Exercises

1. If one of the parallel sides of a trapezoid is double the other, the diagonals trisect each other.

2. In Fig. 404, ABC is any inscribed triangle. CN is perpendicular to AB and CD is a diameter. Prove that:
(*a*) $CB \cdot AN = CN \cdot DB$; (*b*) $AC \cdot NB = CN \cdot AD$;
(*c*) $AC \cdot CB = CD \cdot CN$.

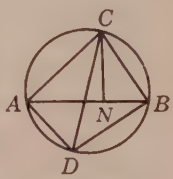


FIG. 404

3. If two circles are tangent externally and a segment is drawn through the point of contact terminated by the circles, the chords intercepted in the two circles have the same ratio as the radii.

4. In Fig. 405, ABC is an isosceles triangle. $\angle 1 = \angle 2$. Prove AC a mean proportional between AB and AD.

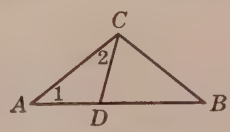


FIG. 405

5. In Fig. 406, ABC is any inscribed triangle. $\angle 1 = \angle 2$. Prove that $AC \cdot RB = RC \cdot AT$ and that $AC \cdot TB = TC \cdot AR$.

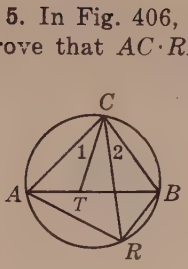


FIG. 406

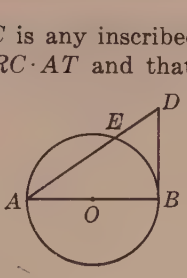


FIG. 407

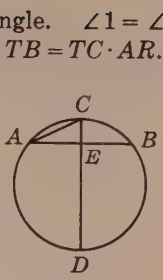


FIG. 408

6. In Fig. 407, AB is a diameter of $\odot O$. BD is tangent to the circle at B. AD is any line from A cutting the circle at E and the tangent at D. Prove that AB is a mean proportional between AE and AD.

7. In Fig. 407, draw BE. Prove that BE is a mean proportional between AE and ED.

8. In Fig. 408, CD is a diameter perpendicular to chord AB. Prove that AC is a mean proportional between CE and CD.

9. In Fig. 408, prove that AE is a mean proportional between CE and ED.

10. Prove $CE \cdot ED = AE \cdot EB$ (Fig. 408).

11. In Fig. 409, $\odot O$ and X intersect at points A and B. AD and AC are tangent to $\odot X$ and O respectively. Prove AB a mean proportional between DB and BC.

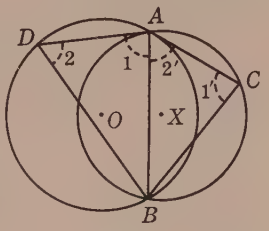


FIG. 409

12. In Fig. 410, ABC is an isosceles triangle inscribed in $\odot O$. Any line is drawn from C cutting AB at E and the circle at D. Prove that $\overline{AC}^2 = CD \cdot CE$.

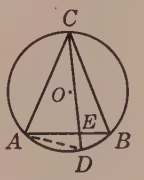


FIG. 410

13. Prove Ex. 12 when CE cuts AB extended.

14. In Fig. 411, AB is a diameter of $\odot O$. BC and AD are tangents at the ends of the diameter. If AC and BD intersect on the circle at E, prove that $\overline{AB}^2 = BC \cdot AD$.

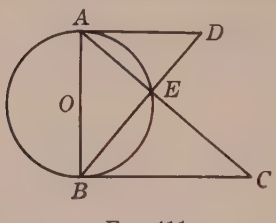


FIG. 411

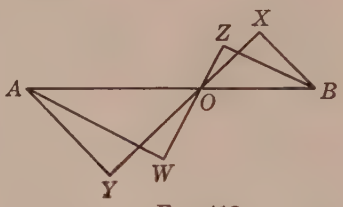


FIG. 412

15. In Fig. 412, O is an arbitrary point in AB. XY and ZW are any two lines through point O. If AY and BX are perpendicular to XY, and AW and BZ perpendicular to ZW, prove that $\dfrac{AY}{BX} = \dfrac{AW}{BZ}$.

16. In Fig. 413, $ABCD$ is a parallelogram and Y is an arbitrary point in DC extended. Prove that $\overline{AO}^2 = OX \cdot OY$.

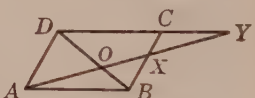


FIG. 413

17. In Fig. 414, ABC is any triangle. $AX = BY$. $XW \parallel BC$ and $YZ \parallel AC$. Prove that $WZ \parallel AB$.

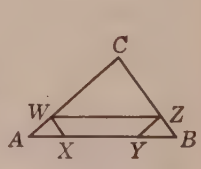


FIG. 414

18. If, in Fig. 414, $AX = BY$, $WX \parallel CB$, and $WZ \parallel AB$, is $YZ \parallel AC$? Give proof.

19. In Fig. 415, the circles are tangent at X. AB and CD are drawn through the points of tangency, meeting the circles as shown. Prove that $AX \cdot XD = CX \cdot XB$.

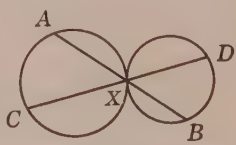


FIG. 415

20. Prove Ex. 19 if the circles are tangent internally.

B Numerical Exercises

1. In Fig. 416, if $AX = 12$, $DX = 4\frac{4}{5}$, and $CX = 20$, find BX.

2. In Fig. 416, if $AB = 19$, $BX = 9$, and $CX = 15$, find AX and DX.

3. In Fig. 416, if $AB = 24$, $DX = 14$, and $CX = 10$, find AX and BX.

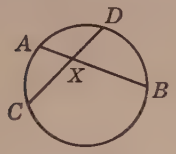


FIG. 416

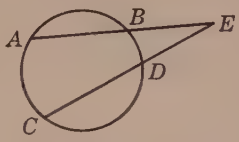


FIG. 417

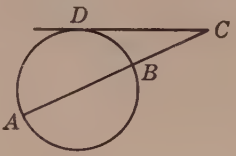


FIG. 418

4. In Fig. 417, if $AE = 15$, $BE = 7$, and $CE = 12$, find DE and CD.

5. In Fig. 417, if $BE = 9$, $CE = 21$, and $DE = 6$, find AE.

6. In Fig. 417, if $CE = 12$, $AB = 13$, and $DE = 4$, find BE.

7. In Fig. 418, if $BC = 4$ and $DC = 8$, find AC.

8. In Fig. 418, if AC passes through the center of the circle and if the radius is $7\frac{1}{2}$ and $DC = 10$, find BC.

9. In Fig. 418, if AC passes through the center of the circle and if $BC = 4$ and $DC = 12$, find the radius.

10. Show that a carpenter's steel square may be used to solve problems in proportion. Fig. 419 shows a steel square graduated to half-inches. ACD is a frame made of two pieces of wood hinged at C. AC can slide on the long arm of the square. Find a number x so that $\frac{12}{16} = \frac{9}{x}$. Place ACD so that the outer edge of CD is on 6 on the long arm (12 half-inches) and $4\frac{1}{2}$ on the short arm. Without changing the angle of adjustment, move the frame higher until the outer edge of CD passes through 8 on the long arm. How is the value of x found? Why?

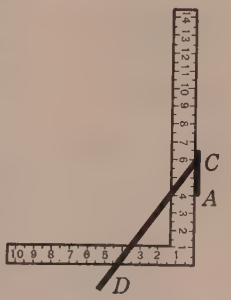


FIG. 419

THEOREM 69

304. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse,

A. Either leg is a mean proportional between the whole hypotenuse and the segment adjacent to that leg.

B. The perpendicular is a mean proportional between the segments of the hypotenuse.

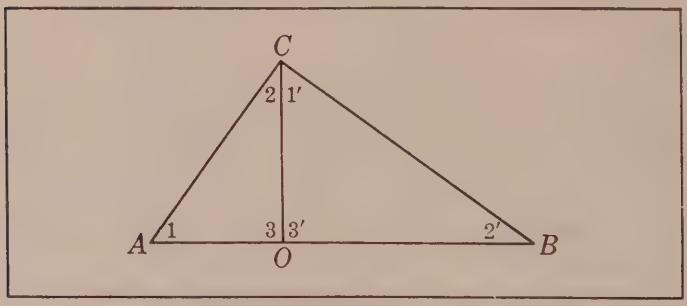


FIG. 420

A. Hypothesis: ABC is a right triangle with $\angle C$ a right angle. $CO \perp AB$ from C.

Conclusion: AC is a mean proportional between AO and AB.

Analysis:

1. To prove AC a mean proportional between AO and AB, prove $\overline{AC}^2 = OA \cdot AB$.
2. To prove $\overline{AC}^2 = OA \cdot AB$, prove $\frac{AO}{AC} = \frac{AC}{AB}$.
3. Use $\triangle AOC$ and $\triangle ABC$ and prove $\angle 1 = \angle 1$, $\angle 3 = \angle C$, and $\angle 2 = \angle 2'$.

Exercise. Give analysis and proof for proving CB a mean proportional between OB and AB.

B. Hypothesis: $\triangle ABC$ is a right triangle with $\angle C$ a right angle. $CO \perp AB$ from C.

Conclusion: CO is a mean proportional between AO and OB.

Analysis:

1. To prove CO a mean proportional between AO and OB, prove $\overline{CO}^2 = AO \cdot OB$.
2. To prove $\overline{CO}^2 = AO \cdot OB$, prove $\frac{AO}{CO} = \frac{CO}{OB}$.
3. Use $\triangle AOC$ and $\triangle BOC$ and prove $\angle 3 = \angle 3'$, $\angle 1 = \angle 1'$, $\angle 2 = \angle 2'$.

COR. **The perpendicular from the vertex of the right angle of a right triangle to the hypotenuse divides the triangle into two triangles similar to the whole triangle and to each other.**

C Exercises

The letters in the next five exercises refer to Fig. 420. Apply Th. 69, A and B.

1. If $AB = 24$ and $AO = 6$, find CO, AC, and CB.
2. If $CO = 18$ and $AO = 27$, find BO and AC.
3. If $AB = 30$ and $AO = 10$, find CO, AC, and CB.
4. If $AO = 12$ and $BO = 15$, find AC, CO, and CB.
5. If $AC = 15$ and $AO = 9$, find AB, CO, and CB.

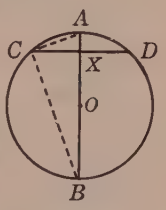


FIG. 421

6. In Fig. 421, AB is a diameter perpendicular to $\overline{CD}$. Show that Th. 69 applies. Find the radius of the circle if $CD = 18$ and $AX = 3$.
7. In Fig. 421, find CD if $AB = 27$ and $AX = 8$.

NOTE. For the method of finding square roots by division, see page 424. See also the table of square roots on page 430. In many cases the table can be used if the number is reduced by the method given on page 425, §418.

Problem 16

305. To construct a mean proportional between two given segments.

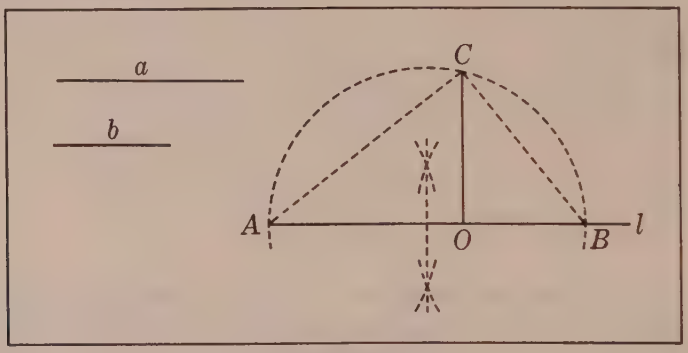


Fig. 422

Given segments a and b.

To construct a mean proportional between a and b.

Directions:

1. On the indefinite line l lay off $AO = a$ and $OB = b$.
2. On AB as diameter draw a semicircle.
3. At O construct a perpendicular to AB meeting the circle at C.
4. CO is the mean proportional between a and b.

Proof: Draw AC and CB and show that Th. 69B, p. 286, applies.

Ex. 1. Construct a mean proportional between segments of 1 cm. and 2 cm. Measure the result in centimeters and compare with the $\sqrt{2}$ as given in the table on page 430. Can you find a segment whose length is $\sqrt{3}$ cm.? $\sqrt{6}$ cm.?

Ex. 2. (*Optional.*) Construct a mean proportional to the two segments given in Fig. 422 by applying Th. 69A, p. 286, instead of Th. 69B. Compare your result with that in Fig. 422.

THEOREM 70*

306. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

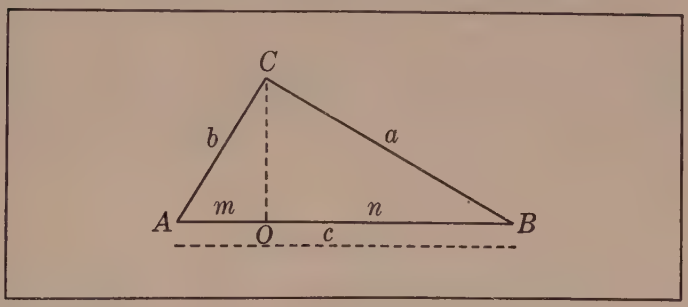


FIG. 423

Hypothesis: ABC is any right triangle with $\angle C$ a right angle, c the hypotenuse, and a and b the legs.

Conclusion: $a^2+b^2=c^2$.

Analysis and construction:

1. To prove $a^2+b^2=c^2$, find a value for a^2 and a value for b^2 and add.
2. Draw a perpendicular from C to AB and find a^2 and b^2 in terms of AB and its segments.

Outline of proof. Give all reasons.

1. $a^2=cn$.
2. $b^2=cm$.
3. $a^2+b^2=cn+cm$.
4. $a^2+b^2=c(n+m)$ (see §307).
5. $a^2+b^2=c^2$.

307. The Addition Theorem. The sum of numbers that have a common factor is the common factor multiplied by the sum of the coefficients.

NOTE. Th. 70 is one of the most important theorems of geometry and one of those most frequently used. More than one hundred proofs are known. It is called the **Pythagorean Theorem.** Pythagoras, a Greek, is supposed to have given a general proof. The nature of the proof that Pythagoras gave is not known, but the fact for a special case may have been known much earlier. Pythagoras settled in Crotona, in southern Italy, where he founded a brotherhood, the members of which were pledged to secrecy. They spent their time in the study of philosophy, ethics, and mathematics and discovered many important theorems in geometry. Pythagoras died about 501 B.C. (see note, pp. 113–114).

C Exercises†

1. Find the hypotenuse of a right triangle if the legs are: (*a*) 5 and 12; (*b*) 3 and 4; (*c*) 8 and 15; (*d*) 4 and 8; (*e*) 6 and 6.

2. Find one leg of a right triangle if the hypotenuse is 17 and one leg is 15.

Solution: Write the equation $x^2+\overline{15}^2=\overline{17}^2$.
Then $x^2+225=289$. Solve for x.

3. Find one leg of a right triangle if the hypotenuse and the other leg are respectively: (*a*) 41 and 40; (*b*) 16 and 8; (*c*) 9 and 3; (*d*) 10 and 6.

4. What is the diagonal of a rectangle if the sides are 25 ft. and 60 ft.?

5. The diagonal of a rectangle is 50 ft. and one side is 14 ft. Find the other side.

6. Find the altitude of an isosceles triangle if the base is 12 and the equal sides are 18. (See §140, p. 123.)

7. Find one leg of an isosceles triangle if the altitude is 18 and the base 12.

8. A chord of a circle is 32 in. and is 8 in. from the center. Find the radius of the circle.

†*To the teacher:* Half of these exercises may be enough for most classes.

9. In a circle whose radius is 36 in. a certain chord is 12 in. from the center. Find the length of the chord.

10. The length of a tangent from a certain point to a circle is 24 in. The point is 25 in. from the center of the circle. Find the radius.

11. A certain point is 24 in. from the center of a circle whose radius is 12 in. Find the length of the tangent from this point to the circle.

12. How much is saved by walking diagonally across a rectangular lot whose sides are 150 and 200 ft.?

13. How long must a rope be to reach from the top of a 50-ft. flagpole to a point on the ground 12 ft. from the base of the pole?

14. The bottom of a 26-ft. ladder is 10 ft. from the wall. How high is the top of the ladder from the ground?

15. One side of a square is 12. Find the sides of an isosceles triangle formed by joining the mid-point of one side to the opposite vertices.

16. One side of a square is 14. Find the sides of an isosceles triangle formed by joining one vertex to the mid-points of the sides not passing through that vertex.

17. The diagonals of a rhombus are 24 and 70. Find one side of the rhombus (see p. 135, C Ex. 3).

18. In a circle whose radius is 18 in. a certain chord is 28 in. Find the distance of the chord from the center.

19. Find the diagonal of a square whose side is 8 ft.

20. If the diagonal of a square is 18 ft., find one side.

21. Find the altitude of an equilateral triangle if one side is 12 ft.; 15 ft. (see §140, p. 123).

22. Find one side of an equilateral triangle if the altitude is 14 ft.; 15 ft.

B Exercises

1. The legs of a right triangle are 10 in. and 24 in. Find the hypotenuse, the altitude on the hypotenuse, and the segments of the hypotenuse made by the altitude.

2. The radii of two concentric circles are 9 and 15 respectively. Find the length of a chord of the outer circle which is tangent to the inner.

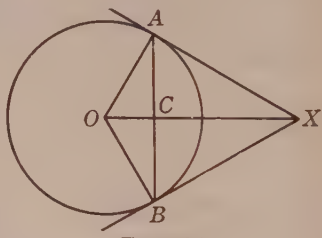


FIG. 424

3. In Fig. 424, AX and BX are tangent to the $\odot O$ from point X. AB is the chord joining the points of contact. If AX and BX are each 24 in. and the radius is 7 in., find OX and AB.

4. In Fig. 424, if $AX = 8$ and $OX = 10$, find OA and AB.

5. In Fig. 424, if $AO = 15$ and $OX = 39$, find AX and AB.

6. In Fig. 424, if $AO = 5$ and $AB = 8$, find AX and OX.

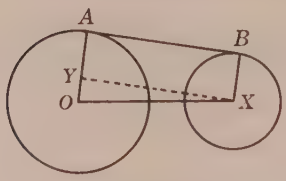


FIG. 425

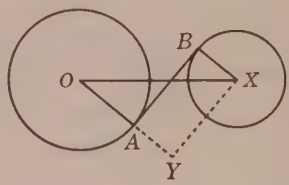


FIG. 426

7. Fig. 425 shows two circles with radii 12 ft. and 8 ft. respectively. If the distance between the centers is 26 ft., find the length of the common external tangent.

Suggestion. Draw XY from $X \perp OA$ and prove $XY = AB$.

8. Fig. 426 shows the same two circles with $OX = 26$ ft. Find the length of the common internal tangent AB.

Suggestion. Draw XY from $X \perp OA$ extended. Prove $XY = AB$.

9. In the middle of a pond 10 ft. square grew a reed. The reed projected one foot above the surface of the water. When blown aside by the wind, its top reached to the mid-point of a side of the pond. How deep was the pond? (An old Chinese problem.)

10. The radius of a circle is 48 ft. Find the distance between two parallel chords which are 72 ft. and 36 ft. if the chords are on the same side of the center.

11. What would be the answer to Ex. 10 if the chords are on opposite sides of the center?

12. The length of the common chord of two intersecting circles is 32. The radii are 65 and 34 respectively. Find the distance between the centers (see §207, p. 194).

13. The distance between the centers of two equal intersecting circles is 18. If the radii are 41, find the length of the common chord.

14. Fig. 427 shows two concentric circles. AB cuts both circles. Find $\overline{AB}$, $\overline{CD}$, and AC, if the radii are 29 and 101 respectively and the distance of $\overline{AB}$ from the center is 20.

FIG. 427

15. If, in Fig. 427, $AB = 70$ and $CD = 10$ and the distance of AB from the center is 12, find the radii.

16. If, in Fig. 427, $AB = 162$, $OA = 82$, and $OD = 41$, find the distance of the chord from the center and find CD.

17. Find the altitude of an isosceles trapezoid if the parallel sides are 40 in. and 58 in. respectively and the non-parallel sides are 41 in.

Suggestion. Construction lines are necessary.

18. The parallel sides of an isosceles trapezoid are 60 and 92 respectively. If the altitude is 63, find the non-parallel sides.

19. The altitude of an isosceles trapezoid is 21 and the non-parallel sides are each 29. If the lower base is 90, find the upper base.

20. The chords drawn from a point on a circle to the ends of a diameter are 14 in. and 48 in. respectively. Find (*a*) the diameter of the circle; (*b*) the distance of each chord from the center; (*c*) the distance of the point from the diameter.

FIG. 428

21. In Fig. 428 is shown a diagram of the roof of a barn. $DE \perp AF$. The rafters AD and FD make equal angles with DE. If $AE = 26$ ft., $AD = 30$ ft., and $DC = 13$ ft., find CB. $CB \perp AF$.

A Exercises

1. *Prove that if one side of a square is b, the diagonal is* $b\sqrt{2}$.

2. *Prove that if the diagonal of a square is d, one side is* $\frac{1}{2}d\sqrt{2}$.

3. *Prove that if one side of an equilateral triangle is b, the altitude is* $\frac{1}{2}b\sqrt{3}$.

4. *Prove that if the altitude of an equilateral triangle is h, one side is* $\frac{2}{3}h\sqrt{3}$.

5. Two parallel chords in a circle are one inch apart. Find the radius of the circle if the chords are 8 and 6 in. long respectively. Is there more than one solution of this problem?

Suggestion. Find two expressions each equal to the square of the radius of the circle and form an equation. Solve the equation.

6. Find the shortest path that an insect can take (without flying) from one corner of a room to the diagonally opposite corner if the room is 15 ft. long, 12 ft. wide, and 10 ft. high.

‡7. Ovals are of frequent use in landscape gardening and other branches of engineering. Agreeable ovals may be laid out as follows (Fig. 429): Let AB be the total length of the desired oval. Lay off on AB two equal intersecting circles with any radius. Draw the common chord CC' and extend CC', making $C'D$ equal to a diameter of the circles. From D draw lines through the centers of the circles O and O' intersecting the circles at X and Y. With D as center and DX as radius draw $\overset{\frown}{XY}$. In a similar manner draw $\overset{\frown}{ZW}$. Show that the circles are tangent at X. If $AO' = \frac{1}{3}AB$ and $AB = 15$ ft., find DD'.

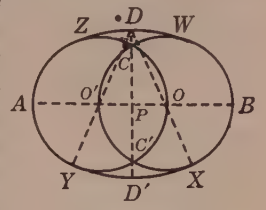


FIG. 429

8. In Fig. 430, arcs AC and BC are drawn with AB as radius and B and A as centers respectively. The circle O is tangent to AC, CB, and to the semicircle constructed on AB as diameter. If $AB = s$ and the radius of $\odot O$ is r, find r in terms of s and construct the figure.

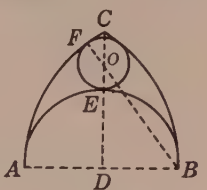


FIG. 430

Suggestion. $OB = s - r$, $OD = \frac{1}{2}s + r$, $DB = \frac{1}{2}s$, $\overline{OB}^2 = \overline{OD}^2 + \overline{DB}^2$. Make the substitutions and solve the equation.

9. Fig. 431 shows an isosceles right triangle with its inscribed circle. X is the center of the circle. Prove $\dfrac{CX}{XO} = \dfrac{\sqrt{2}}{1}$.

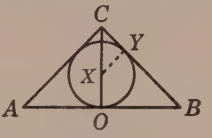


FIG. 431

10. In Fig. 431 find the length of OX if $AB = 4$.

11. Prove the Pythagorean theorem by means of Fig. 432.

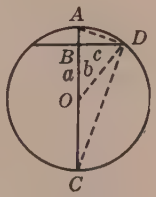


FIG. 432

Suggestion. Let a represent the distance OB, b represent the radius of $\odot O$, and c represent the half chord perpendicular to AC at B. Prove that $(b-a)(b+a) = c^2$, or $b^2 - a^2 = c^2$.

‡ Depends upon optional work.

12. Two equal circles intersect in such a manner that the common chord is equal to the segment joining the centers. If the common chord is 6 in., find the radius of the circles and the width of the lens-shaped figure formed.

13. Find the locus of points from which the distances to two given intersecting lines are in a given ratio.

14. Find by experiment the locus of points whose distances from two given points are in a given ratio.

15. Show that a^2+b^2, a^2-b^2, and $2ab$ can represent the numerical values of the sides of a right triangle.

16. Substitute integers for a and b in the formula given in Ex. 15 and obtain sets of numbers which may be the sides of a right triangle.

Illustration. Let $a=2$ and $b=1$, then $a^2+b^2=5$, $a^2-b^2=3$, $2ab=4$. 3, 4, and 5 may be the sides of a right triangle.

NOTE. Since geometry was first studied, men have been interested in finding integers that would represent the sides of a right triangle. The fact that a triangle whose sides are 3, 4, and 5 is a right triangle was probably known to the Egyptians more than fifteen hundred years before the time of Pythagoras. The Egyptians used ropes knotted at equal distances. These were stretched about three poles so as to form a right triangle. The Egyptians called the men who knew how to use these ropes *rope-stretchers*. Surveyors use similar methods today. How many knots must there be in the rope and how is it used to construct a right triangle?

The Chinese and the Hindus probably knew about the right triangle whose sides are 3, 4, and 5 at a very early date.

The rule given in Ex. 15 for finding integers to represent the sides of a right triangle was given by an Englishman, Masères, who died in 1824. But rules similar to this were known to the ancient Greeks. Pythagoras, Euclid, and Plato each had such a rule. What do you know about Pythagoras and Euclid? Plato (427–347 B.C.) was a philosopher. That he had a high opinion of mathematics is shown by the fact that he had written over the door of his school a sign, "Let none ignorant of geometry enter my door." He insisted on logical thinking and clear, exact statements in geometry.

B Miscellaneous Exercises — Applications of Equal Ratios

1. Fig. 433 shows proportional compasses used to enlarge or reduce drawings to scale. The lengths of the legs are adjusted by means of the screw at O. What ratios are equal? Why?

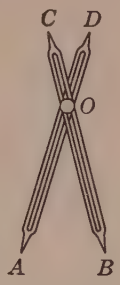


FIG. 433

2. If, in Fig. 433, $OA = 6$ in. and $OD = 2$ in., what is the ratio of AB to CD? Why?

3. We know that at any given moment the sun's rays make the same angle with the ground, at any particular place; that is, if, in Fig. 434, AC is the shadow of the tree AB and XZ is the shadow of the post XY at the same minute, $\angle C = \angle Z$. Prove $\triangle ABC \sim \triangle XYZ$.

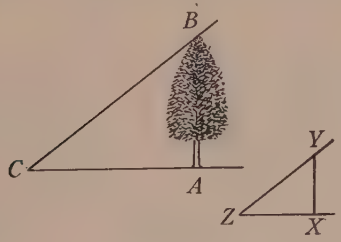


FIG. 434

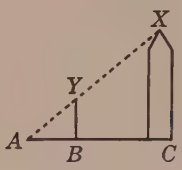


FIG. 435

4. If the shadow of the tree in Ex. 3 is 60 ft. at the same time that the shadow of a 5-ft. pole is 8 ft., find the height of the tree.

5. A man sets up a 5-ft. pole at B (Fig. 435). Then, lying flat on the ground at A, he adjusts his position so that he can just see the top of the tower on line with the top of his pole. If his eye is 5 ft. 5 in. from B and 100 ft. from C, find the height of the tower.

6. What possibilities of error are there in work like that stated in Exs. 4 and 5?

7. A flagpole is situated on top of a building. If the shadow of the building is 88 ft. and of the building and pole together 101¾ ft. at the same time that an 8-ft. post casts a shadow 5½ ft. long, find the height of the building and the length of the pole.

8. A man wishes to find the distance across a pond (Fig. 436). He sets up stakes at B and E so that B can be seen from E. He then sets up a stake C in line with B and E and lays out line CDA, placing the stake at D so that $DE \parallel AB$. If $AB = 150$ ft., $DE = 100$ ft., and $CE = 180$ ft., find the distance from B to E.

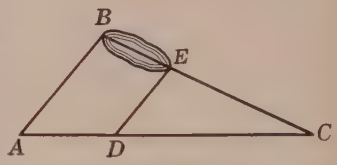


FIG. 436

9. Show that the distance from A to an inaccessible point B (Fig. 437) may be obtained by the following method: Make $AC \perp AB$. Take C, any point on AC from which B is visible. Make $CD \perp CB$. Find E, the point at which CD and BA would intersect. What lines must be measured?

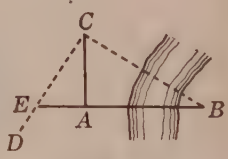


FIG. 437

10. Show how the measurement suggested in Ex. 9 may be performed practically by the aid of a pole and a carpenter's steel square.

NOTE. In Ex. 9 stakes are set up at the points indicated and the figure laid out on the ground. In Ex. 10 the figure is set up in a vertical plane.

11. The span of a circular arch is 120 ft. If the radius of the circle of which it is a part is 75 ft., find the height of the middle of the arch.

12. If the cross-section of a street surface is the arc of a circle, the distance from curb to curb is 30 ft., and the rise of the center of the street above the gutter is 7 inches, what is the radius of the circle?

13. A man wishes to find the height of a tower. He places a mirror on the ground at X and walks backward until he sees the top of the tower at a point marked on the mirror. If $AX = 4$ ft., $CX = 28$ ft., and the man's eye is 5 ft. 9 in. from the ground, find the height of the tower (Fig. 438). A indicates the point at which the man is standing and BA the height of the man's eye from the ground.

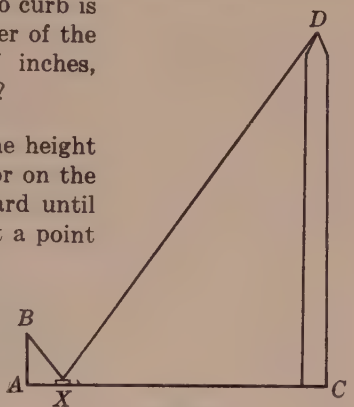


FIG. 438

14. Show that an inaccessible distance AB (Fig. 439) may be obtained thus: Make $CA \perp AB$ and extend AC until CD is some convenient part of AC ($\frac{1}{2}$ or $\frac{1}{3}$). Make $DE \perp AD$. Find E, the point at which BC and DE would intersect. What measurement must be taken? Is it necessary that AB and DE be $\perp AD$? Why?

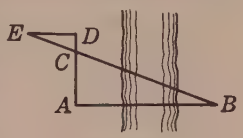


FIG. 439

15. Before the invention of the telescope an instrument called the cross-staff was sometimes used to measure inaccessible heights and distances. The cross-bar CD was made to slide up and down the staff BA (Fig. 440). Show how this instrument could be used to find the width of the stream R. How could it be used to find the height of a steeple?

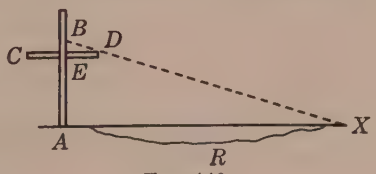


FIG. 440

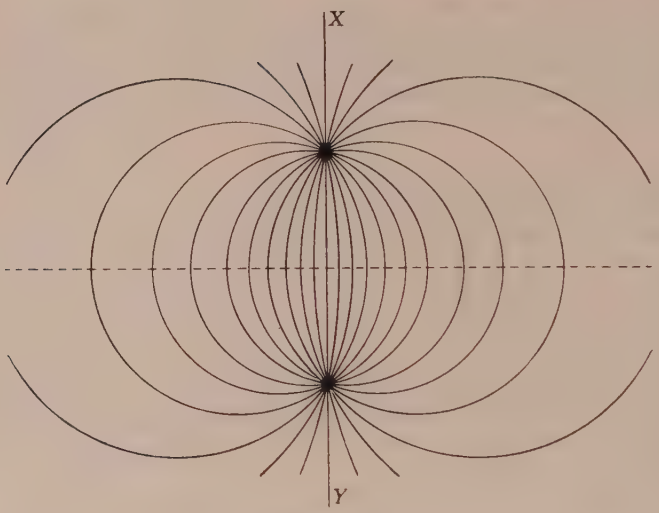


FIG. 441

A Exercises

1. Draw Fig. 441, putting in 3 or 4 circles on each side of XY. Prove that tangents to these circles from any point in XY are equal.

2. Mark the center of one of your circles O. Let the tangent to circle O from X meet circle O at A. Prove that OA would be tangent to a circle drawn with X as center and XA as radius.

NOTE. If two circles cut each other in such a way that tangents at the point of intersection are perpendicular to each other, the circles are said to cut each other at right angles.

3. Draw a series of circles that will cut all the circles that you drew in answer to Ex. 1 at right angles.

UNIT VI

SOLUTION OF RIGHT TRIANGLES (*Optional*)

308. A Greek, Thales, who lived from about 640 to 548 B.C., introduced the study of geometry from Egypt into Greece. It is related that when in Egypt he astonished the Egyptians by his ability to measure the height of the pyramids by means of his walking stick and the shadows of the stick and of the pyramids.

309. You have already learned two methods of measuring inaccessible distances. The first used congruent triangles (see p. 55 and note, p. 56); the second, similar triangles (B Miscellaneous Exercises, p. 297). A third method based on similar triangles is now to be presented.

310. If two right triangles have an acute angle of one equal to an acute angle of the other, the ratio of any two sides of one is equal to the ratio of the two corresponding sides of the other.

The proof for the above is left to you.

311. The sine ratio. The triangles in Fig. 442 are right triangles with $\angle A$ common. They are similar.

The ratio $\frac{CB}{AB}$ is the same for all right triangles that contain $\angle A$. Why? This ratio has been given a special name and is called the **sine of $\angle A$** or, for short, **sin A.** The sines of angles for every degree from 1 to 90 have been computed and placed in tables for general use (see p. 431).

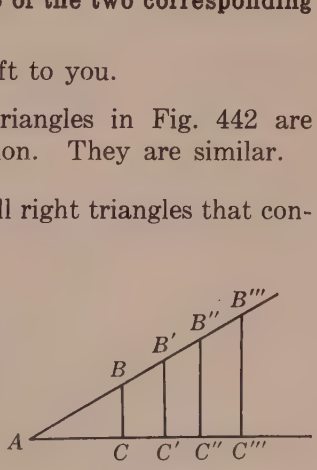


FIG. 442

Ex. 1. Construct with the protractor a right triangle with $\angle A$ equal to 50° (Fig. 443). Find a and c in millimeters and compute sin 50°. Compare the result obtained with that given in the table, p. 431.

NOTE. In Fig. 443, the triangle is lettered so that side a is opposite $\angle A$, side b is opposite $\angle B$, side c is opposite $\angle C$. Angle C is the right angle. For convenience in the discussion that follows, the right triangles will be lettered in this way.

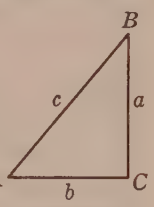


FIG. 443

Ex. 2. Work Ex. 1 for triangles with angles of 40°, 32°, 68°.

Ex. 3. Construct a right triangle with a and c in the ratio of 3 to 5. Measure $\angle A$ with the protractor and find in the table the angle whose sine is nearest .600.

Ex. 4. If $a=23$ and $c=48$, find $\angle A$.

Solution: $\sin A=\frac{a}{c}=\frac{23}{48}$.

$\sin A=.479$ (by division).

$\therefore \angle A=28°$ about (from table).

Ex. 5. If $a=35$ and $c=47$, find $\angle A$.

Ex. 6. If $\angle A=19°$ and $c=16$, find a.

Solution: $\sin A=.326$ (from table).

$\sin A=\frac{a}{c}=\frac{a}{16}$.

$\therefore \frac{a}{16}=.326$. Solve for a.

Ex. 7. If $\angle A=41°$ and $a=86$, find c.

Solution: $\sin A=.656$ (from table).

$\sin A=\frac{a}{c}=\frac{86}{c}$.

$\therefore \frac{86}{c}=.656$.

$\therefore 86=.656\,c$. Solve for c.

Ex. 8. If $\angle A=29°$ and $c=45$, find a.

Ex. 9. $\angle A=31°$ and $a=54$, find c.

312. The cosine ratio. The ratio of $\frac{AC}{AB}$ is the same for all right triangles that contain the same acute $\angle A$. Why? This ratio is called the **cosine of $\angle A$** or, for short, **cos A** (Fig. 444).

Ex. 1. If $c = 62$ and $b = 26$, find $\angle A$.

Ex. 2. If $c = 86$ and $b = 35$, find $\angle A$.

Ex. 3. If $\angle A = 38°$ and $c = 110$, find b.

Ex. 4. If $\angle A = 56°$ and $b = 94$, find c.

Ex. 5. If $\angle A = 61°$ and $c = 216$, find b.

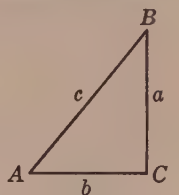


FIG. 444

313. The tangent ratio. The ratio of $\frac{CB}{AC}$ is the same for all right triangles that contain the same acute $\angle A$. Why? This ratio is called the **tangent of $\angle A$** or, for short, **tan A** (Fig. 444).

Ex. 1. If $a = 75$ and $b = 60$, find $\angle A$.

Ex. 2. If $a = 53$ and $b = 104$, find $\angle A$.

Ex. 3. If $\angle A = 22°$ and $b = 39$, find a.

Ex. 4. If $\angle A = 67°$ and $a = 42$, find b.

Ex. 5. If $\angle A = 27°$ and $b = 112$, find a.

314. The definitions of the ratios. These three ratios are called functions of the angle because they depend upon the size of the angle for their values. They may be defined as follows:

$$\sin A \text{ or } \frac{a}{c} = \frac{\text{side opposite } \angle A}{\text{hypotenuse}}$$

$$\cos A \text{ or } \frac{b}{c} = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}}$$

$$\tan A \text{ or } \frac{a}{b} = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A}$$

A Exercises — Important Special Cases

1. Construct an equilateral triangle and its altitude. Prove without consulting the table that $\sin 30° = \cos 60° = \frac{1}{2}$.

2. *From the results obtained in Ex. 1 show that in a 60°–30° right triangle the shorter leg is one-half the hypotenuse.*

3. Prove the converse to Ex. 2.

4. Let one side of an equilateral triangle be b. Show without consulting the table that $\sin 60° = \cos 30° = \frac{1}{2}\sqrt{3}$. What is $\tan 60°$? $\tan 30°$?

5. Construct an isosceles right triangle. Show without consulting the table that $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$ and that $\tan 45° = 1$.

315. Angles of elevation and depression. If you are standing at A (Fig. 445), the angle A between the horizontal AC and the line of sight AB is called the **angle of elevation.** It is the angle through which the telescope in the instrument is elevated in finding $\angle A$. Similarly $\angle 2$ is called the **angle of depression.** Why? Note that $\angle 1 = \angle 2$.

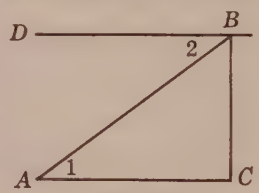


FIG. 445

B Exercises

NOTE. Before attempting any of the following exercises, draw the figure carefully, write the data given on the figure (see Ex. 1), and decide which ratio is to be used.

1. It is desired to find the height of a certain chimney. The distance AC is measured and found to be 100 ft. $\angle A$ is found to be 36°. What is the height of the chimney (Fig. 446).

Suggestion. Use the tangent ratio.

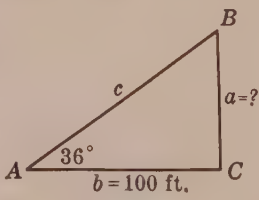


FIG. 446

2. At a distance of 75 ft. from the foot of a tree the angle of elevation of the top is found to be 32°. How high is the tree?

3. A memorial flagstaff in a park, casts a shadow 47 ft. long when the angle of elevation of the sun is 60°. How high is the flagstaff?

4. A ladder is 20 ft. long. How far from a building must the foot be placed so that it will make an angle of 70° with the ground?

5. How far up the side of the building will the ladder mentioned in Ex. 4 reach?

6. Find the altitude of an equilateral triangle whose side is 50.

7. The vertex angle of an isosceles triangle is 50°. The altitude is 10 ft. Find the three sides.

8. A mountain road makes an angle of 5° with the horizontal. If a man goes one and one-half miles up the road, how high has he risen?

9. In Fig. 447, ABC is a right triangle with $\angle C$ a right angle. $CD \perp AB$. If $AC = 15$ and $\angle A = 40°$, find AD, CD, CB, and DB.

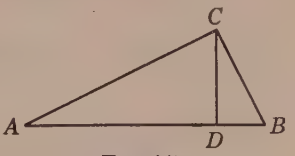


FIG. 447

10. If, in Fig. 447, $CD = 10$ and $\angle B = 63°$, find the other segments in the figure.

11. If, in Fig. 447, $AB = 25$ and $BC = 15$, find AC and $\angle A$ and $\angle B$.

12. A boy 5 ft. 4 in. tall casts a shadow 10 ft. long. What is the angle of elevation of the sun?

13. A boy is flying his kite with a string 500 ft. long. He observes the angle of elevation of his kite to be 75°. How high is the kite?

NOTE. Ex. 13, of course, assumes that the string is straight, which is never really true. Will the height found be too great or too small? Why?

14. Two boys on one side of a river at C wish to find the distance across a river from A to C (Fig. 448). If there is a tree at A, how are the stakes at C and B laid out and what measurements should be taken?

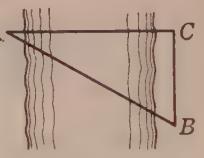


FIG. 448

15. A road rises 6 ft. in a horizontal distance of 100 ft. What is the angle of elevation?

16. How high would a traveler rise in going a mile along the road mentioned in Ex. 15?

17. How far along the road mentioned in Ex. 15 would a traveler pass to rise 400 ft.?

18. Find the sides of an equilateral triangle whose altitude is 50.

19. A man wants a garage to be 40 ft. wide. He also wants the rafters to make an angle of 45° with the horizontal. How long will the rafters be and how high will the ridgepole be above the beam AB? ($\triangle ABC$ is isosceles, Fig. 449).

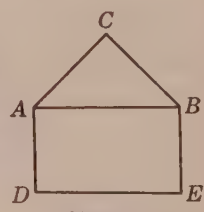


FIG. 449

20. A boy looks out of a window 300 ft. above the street. The angle of depression of an object on the other side of the street is 72°. How wide is the street?

21. A tent is to be made with a 30-ft. center pole and a 9-ft. side wall (Fig. 450). The guy ropes are to make an angle of 25° with the ground. What is the diameter of the tent and the length of the guy rope from the peak of the tent?

FIG. 450

22. If the length of the shadow of a tree is twice the height of the tree, what is the angle of elevation of the sun?

23. The legs of a right triangle are 12 and 16. Find the two acute angles.

24. A boat sails southeast 4 miles. How far is it east from the starting point?

25. A ship sails 6 miles on a course 28° west of south. How far west is it from the starting point?

26. How far south of the starting point is the ship mentioned in Ex. 25?

27. The sides of a rectangle are 20 and 12. What angle does the longer side make with the diagonal? the shorter side?

28. The shadow of a building is observed to be 100 ft. long and at the same time the shadow of a lamp-post 8 ft. high is 5 ft. long. Find the angle of elevation of the sun and the height of the building.

29. The radius of a circle is 20 in. Find the length of a chord whose arc is 50°.

30. A chord of a circle is 8 in. Its central angle is 36°. Find the radius of the circle and the distance of the chord from the center.

Survey Test

1. (*a*) What do you understand by the ratio of two numbers? (*b*) Explain what is meant by the value of the ratio. (*c*) When are two ratios said to be equal?

2. (*a*) What is the fundamental law of ratios? (*b*) Give examples of its use in arithmetic. (*c*) Quote one other axiom frequently used in ratios. (*d*) Give examples of its use.

3. (*a*) Define a proportion. (*b*) Give an example. (*c*) In your illustration name the means; the extremes; the first term; the third term. Define the means and the extremes of a proportion.

4. Obtain a pair of equal products from the proportion $\frac{a}{b}=\frac{x}{y}$. Quote the theorem used.

5. Obtain a pair of equal ratios from the equal products $9\times8=4\times18$. Quote the theorem used.

6. If $\frac{a}{b}=\frac{c}{x}$ and $\frac{a}{b}=\frac{c}{y}$, quote the theorem that proves $x=y$.

7. Explain what is meant by (*a*) mean alternation; (*b*) extreme alternation; (*c*) inversion; (*d*) proportion by addition or composition; (*e*) proportion by subtraction or division.

8. (*a*) Write a proportion using the ratio $\frac{18}{27}$. (*b*) Write it in each of the five ways mentioned in question 7. (*c*) Prove that your proportions are true.

9. (*a*) What do you understand by the measure number of a segment? (*b*) In measuring segments with a given unit what three possibilities arise? (*c*) What kind of number is the measure number in each case?

10. (*a*) What do you understand by the ratio of two segments? (*b*) When are two segments said to be commensurable? incommensurable? (*c*) Name three cases of incommensurable segments.

11. (*a*) What is a rational number? an irrational number? (*b*) Give examples of each. (*c*) What differences do you know between rational and irrational numbers?

12. Draw a triangle with a line parallel to one side, not the base. Write at least two pairs of ratios that are equal in your figure and give the reason in each case.

13. Draw three parallels cutting two transversals. Write at least two pairs of ratios that are equal in your figure and give the reason in each case.

14. (*a*) Quote the theorem you would use to divide a given segment into two parts that shall have the same ratio as two given segments. (*b*) What other construction is based on this theorem?

15. Quote one theorem in proportion that is used to prove lines parallel.

16. What two facts must you prove in order to prove two polygons similar?

17. What do you understand by the ratio of similitude of two similar polygons?

18. What method can be used to prove two triangles similar that cannot be used for other polygons?

19. (*a*) What methods are used to prove two ratios equal? (*b*) Quote in full the theorems or corollaries used. (*c*) Which method did you use most often?

20. How can you prove two products equal?

21. How do you select the proper triangles when you wish to prove two ratios equal by the use of similar triangles?

22. (*a*) How do you select corresponding sides of similar triangles? (*b*) Illustrate by a figure from the book.

Complete each of the three following:

23. If two chords intersect within a circle . . . . . .

24. If two secants intersect without a circle . . . . . .

25. If a secant and a tangent intersect without a circle . . . . . .

26. Define a mean proportional.

27. Give an example from arithmetic of a mean proportional.

28. Quote three cases in which one segment is a mean proportional between two others.

29. (*a*) On what theorem would you base the construction of a mean proportional between two given segments? (*b*) Make the drawing.

30. Quote two theorems from this chapter that are especially concerned with right triangles.

31. Quote the Pythagorean Theorem.

32. Give a practical example of the use of the Pythagorean Theorem.

33. Quote the Addition Theorem. Give an example of its use in algebra; in geometry.

Remedial Study

If you are unable to answer any of the foregoing questions, the following table will tell you what pages to consult for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	253	11	259	22	275
2	254	12	260, 261	23	276
3	254	13	262	24	277
4	254	14	263, 265	25	279
5	255	15	266	26	278
6	255	16	271	28	279, 286
7	255, 256	17	271	29	288
8	255, 256	18	273	30	286, 289
9	257, 258	19	274	31	289
10	258, 259	20	274	33	289
		21	274		

UNIT VII

AREAS OF POLYGONS

316. Uses of areas. To find the area of a polygon, we must measure its surface. This is of much importance in practical geometry. When a roof is to be shingled, its area must be known. Areas must also be found when estimates of various kinds are to be made, such as the quantity of paint needed for a frame house or the lumber required for a floor. The exercises on pages 326 and 327 present problems met by surveyors and the methods used to solve them. Can you think of any other instances in which it is essential that areas be known?

317. Congruent polygons can be made to coincide. They have the same size and shape. Two polygons may have the same size without having the same shape. Such polygons cannot be made to coincide.

Exercise. Construct two congruent right triangles that are not isosceles. Cut them out and place them together in different positions so as to form various polygons all of which cover the same extent of surface but no two of which are congruent. Make careful drawings on cross-section paper.

318. The axioms as applied to congruent figures. Note that the axioms on pages 29 and 30 do not apply to congruent figures. Use the following postulate:

Post. 46. If congruent polygons are added to or subtracted from congruent polygons, the results are polygons that cover the same extent of surface.

Exercise. Turn elsewhere in the book and find illustrations of polygons that (*a*) appear to have been added; read their sum; (*b*) appear to have been subtracted; read their difference.

319. If, then, you wish to prove that two polygons cover the same extent of surface, prove that they are made up of parts congruent in pairs.

Ex. 1. In Fig. 451, $\triangle ABC$ is any triangle with D the mid-point of AC, $DF \parallel AB$, and $BF \parallel AC$. Prove that $\triangle ABC$ and $\square ABFD$ cover the same extent of surface.

Analysis:

I. To prove that $\triangle ABC$ and $\square ABFD$ cover the same extent of surface, prove that they are made up of parts congruent in pairs.

II. $\therefore$ show that $ABED \cong ABED$ and $\triangle DEC \cong \triangle BEF$.

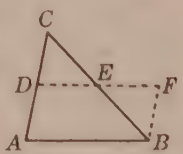


FIG. 451

Ex. 2. Construct a parallelogram that shall cover the same extent of surface as a given trapezoid.

TOPIC 1. MEASUREMENT OF SURFACES OF POLYGONS

320. To **measure the surface** inclosed by the sides of a polygon is to find how many times it contains another surface chosen as a unit of measure.

321. The **area of a polygon** is the measure number of the surface of the polygon.

It is the common practice to use as a unit of surface a square whose side is a unit of length. Thus if the unit of length is an inch, the unit of surface is a square whose side is an inch. It is called a square inch.

While any segment may be used as a unit of length with its corresponding unit of surface, it is most convenient practically to use one of the recognized standard units of length, such as the inch, foot, yard, mile, tenth of an inch, centimeter, etc., with their corresponding units of surface—square inch, square mile, etc.

THEOREM 71

322. The area of a rectangle is the product of its base and altitude.

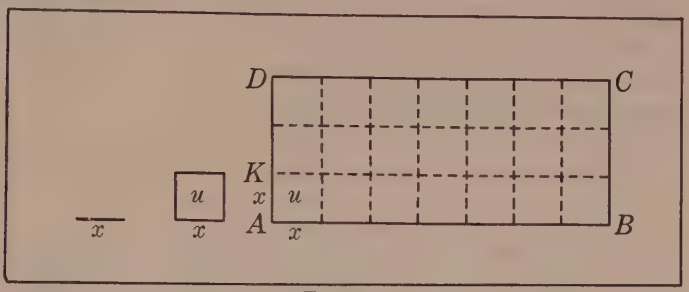


FIG. 452

The meaning of Theorem 71 is that the number of units of surface in a rectangle is equal to the product of the number of units of length in the base, and the altitude.

Hypothesis: $ABCD$ is a rectangle; AB is the base and AD the altitude.

Conclusion: Area $ABCD = AB \cdot AD$.

A. When AB and AD have a common measure.

Proof:

The rectangle can be divided into unit squares.

Suppose that x or AK (Fig. 452) is the common unit of measure of AB and AD. Then the square u, with x as one side, will be the unit of surface. x may be contained in the base AB m times and in the altitude AD n times. By drawing lines through the points of division on AD and AB parallel to the sides, the rectangle is divided into n rows with m unit squares in a row. That is, the unit of surface u is contained in the rectangle mn times. The area is mn unit squares. Since m and n are the measures of AB and AD,

$$\text{area } ABCD = AB \cdot AD.$$

B. When the sides of the rectangle have no common measure.

In this case the rectangle cannot be divided into unit squares as in Case A.

It can be shown that the theorem is true in this case also, but the discussion is too difficult for this course. You may assume it. Thus, if the altitude is $\sqrt{3}$ and the base $\sqrt{5}$, the area is $\sqrt{3}\sqrt{5}$ or $\sqrt{15}$.

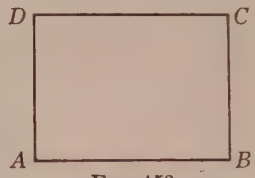


FIG. 453

In such cases the area may be calculated to any degree of approximation, $\sqrt{15} = 3.873$. In practical work approximate lengths are used for both altitude and base. These are found by measuring with a scale showing the standard unit used.

323. Theorem 71 is usually written $A = hb$, where A is the measure number of the surface, b the measure number of the base, and h of the altitude.

C Exercises

1. The base and altitude of a rectangle are $8\frac{3}{4}$ in. and $6\frac{1}{2}$ in. respectively. What is the measure number of its surface?

2. The base and altitude of a rectangle are $3\frac{1}{8}$ in. and $5\frac{1}{2}$ in. respectively. What is its area?

3. From the formula $A = hb$ state a formula for the base and one for the altitude of a rectangle.

4. Find the base of a rectangle if its area is 134 sq. in. and its altitude $5\frac{1}{2}$ in.

5. A rectangle with base 16 ft. has the same area as a square 24 ft. on a side. Find its altitude.

6. Find the side of a square that has the same area as a rectangle 80 ft. by 15 ft.

7. One rectangle is $16\frac{1}{2}$ ft. by 30 ft. Find the altitude of a second rectangle that has the same area if its base is 45 ft.

8. Find the side of a square whose area is equal to the sum of the areas of two squares whose sides are 18 and 81 respectively.

9. Two rectangles each have an area of 720 sq. ft. Find the difference between their altitudes if the bases are 40 and 45 ft. respectively.

B Exercises

1. The area of a square is 576 sq. ft. Find the area of a rectangle with the same perimeter if its base is 32 ft.

2. One side of a rectangle is 36 and its diagonal is 85. Find its area.

3. If the dimensions of a rectangle are 16 and 63, find the area of a square that has the diagonal of the rectangle for one side.

4. The dimensions of one rectangle are 15 ft. and 60 ft. and of a second 25 ft. and 50 ft. Find the base of a rectangle equal to the sum of their areas if its altitude is 40 ft.

5. Fig. 454 shows the cross section of a column. If $AB = BC = CD = a$ and the width is everywhere d, find a formula for its area.

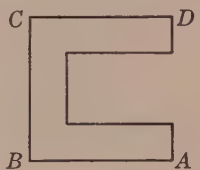


FIG. 454

6. Find the area of a walk 3 ft. 6 in. wide that completely surrounds a lot 300 ft. by 500 ft. The dimensions of the lot are taken on the inside of the walk.

7. The base of a rectangle is 4 ft. more and its altitude 3 ft. less than the sides of a square that has the same area. Find the dimensions of each.

THEOREM 72*

324. The area of a parallelogram is the product of the base and altitude.

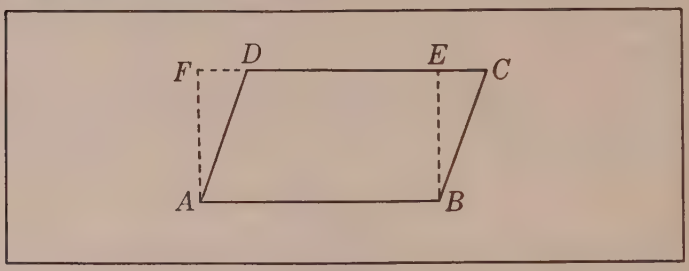


FIG. 455

Hypothesis: $ABCD$ is any $\square$ with base AB and altitude BE.

Conclusion: Area $ABCD = AB \cdot BE$.

Analysis and construction:

1. To prove area $ABCD = AB \cdot BE$, prove that the parallelogram covers the same extent of surface as a rectangle whose sides are AB and BE.
2. $\therefore$ complete the rectangle $ABEF$ and prove that the $\square ABEF$ and the $\square ABCD$ are made up of parts congruent in pairs.
3. $\therefore$ show that $ABED \cong ABED$ and $\triangle ADF \cong \triangle BCE$.

Outline of proof. Give the reasons.

1. rt. $\triangle ADF \cong$ rt. $\triangle BCE$.
2. $ABED \cong ABED$.
3. $\therefore$ $ABEF$ and $ABCD$ cover the same extent of surface.
4. The measure of $ABEF = AB \cdot BE$.
5. $\therefore$ the measure of $ABCD = AB \cdot BE$,
6. or the area of $ABCD = AB \cdot BE$.

Exercise. Prove Th. 72 when E falls between D and F.

THEOREM 73

325. The area of a triangle is one-half the product of the base and altitude.

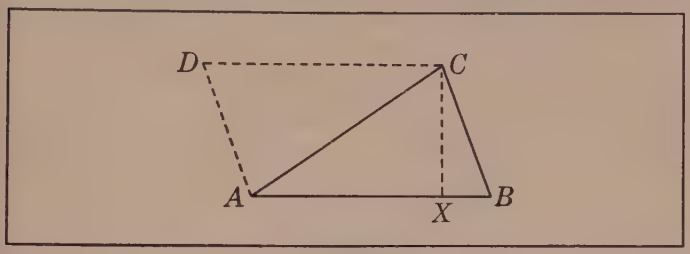


FIG. 456

Hypothesis: ABC is any $\triangle$ with base AB, altitude CX.

Conclusion: Area $ABC = \frac{1}{2}AB \cdot CX$.

Analysis and construction:

1. To prove area $ABC = \frac{1}{2}\ AB \cdot CX$, prove the surface of $\triangle ABC$ is $\frac{1}{2}$ the surface of a parallelogram whose sides are AB and BC.
2. $\therefore$ complete the parallelogram $ABCD$.

Outline of proof. Give all reasons.

1. Surface of $\triangle ABC = \frac{1}{2}$ the surface of $\square ABCD$ (Th. 24, p. 132).
2. The measure of $\square ABCD = AB \cdot CX$.
3. $\therefore$ the measure or area of $\triangle ABC = \frac{1}{2}AB \cdot CX$.

326. Formulas for areas. In §323, p. 314, Th. 71 was stated thus:

For rectangles $A = hb$

where A represents the area, b the measure of the base, and h the measure of the altitude.

Similarly, using the notation above, we have

For parallelograms $A = hb$

For triangles $A = \frac{1}{2}hb$

THEOREM 74*

327. The area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases.

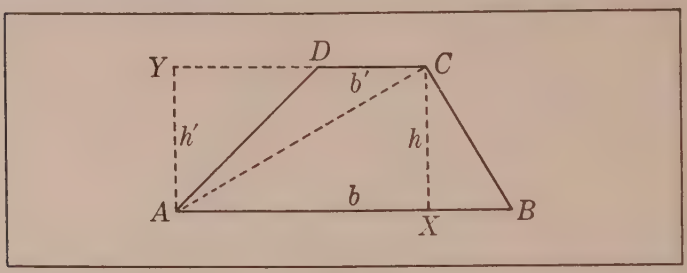


FIG. 457

Hypothesis: $ABCD$ is any trapezoid with base $AB = b$, base $CD = b'$, and altitude $CX = h$.

Conclusion: Area $ABCD = \frac{1}{2}h(b + b')$.

Analysis and construction:

1. To prove area $ABCD = \frac{1}{2}h(b + b')$, divide $ABCD$ into two triangles and add their areas.
2. $\therefore$ draw AC dividing $ABCD$ into $\triangle ABC$ and ADC. Draw AY, the altitude of $\triangle ADC$, using DC as base.

Outline of proof. Give all reasons.

1. Area $ABC = \frac{1}{2}hb$.
2. Area $ACD = \frac{1}{2}h'b'$.
3. $\therefore$ area ABC + area $ACD = \frac{1}{2}hb + \frac{1}{2}h'b'$.
4. But $h = h'$ (§151, p. 132).
5. $\therefore$ area $ABCD = \frac{1}{2}hb + \frac{1}{2}hb'$.
6. $\therefore$ area $ABCD = \frac{1}{2}h(b + b')$ (§307, p. 289).

328. The formula for the area of a trapezoid is

$$\boldsymbol{A = \frac{1}{2}h(b + b')}$$

C Exercises

NOTE. Use the equation form for the exercises below.

1. Find the area of each of the following parallelograms:
   (*a*) $h = 3\frac{1}{2}$; $b = 2$ ft. 4 in. (*b*) $h = 8$ ft. 3 in.; $b = 5$ ft. 4 in.

2. Find the area of each of the following triangles:
   (*a*) $h = 13$ ft.; $b = 6$ ft. (*c*) $h = 14\frac{1}{2}$ ft.; $b = 9\frac{1}{2}$ ft.
   (*b*) $h = 4.3$ ft.; $b = 6.2$ ft. (*d*) $h = 10$ ft. 6 in.; $b = 9$ ft. 2 in.

3. Find the area of each of the following trapezoids:
   (*a*) $h = 12$ in. $b = 21$ in. $b' = 9$ in.
   (*b*) $h = 6\frac{1}{2}$ ft. $b = 15\frac{1}{3}$ ft. $b' = 9\frac{2}{3}$ ft.

4. Find the altitude of a triangle whose base is 13 ft. 4 in. and whose area is 50 sq. ft.

Suggestion. $A = \frac{1}{2}hb$, $\therefore\ 50 = \frac{1}{2}h(13\frac{1}{3})$. Solve for h.

5. Find the base of a triangle whose area is $33\frac{3}{4}$ sq. ft. and whose altitude is 9 ft.

6. Find the altitude of a trapezoid if its area is 72 sq. in. and the bases 12 in. and 20 in. respectively.

7. Find one base of a trapezoid if its area is 345 sq. in., the other base 45 in., and the altitude $11\frac{1}{2}$ in.

8. Find the altitude of a triangle that has the same area as a parallelogram whose base is 6 in. and whose altitude is $8\frac{2}{3}$ in., if the base of the triangle is 13 in.

9. The bases of a trapezoid are 50 and 56. Find its altitude if its area is equal to that of a rectangle 57 by 10.

In the following isosceles triangles, find

10. Altitude and area, if the base = 32 and one leg = 34.

11. Base and area, if the altitude = 99 and one leg = 101.

12. One leg and area, if the altitude = 77 and the base = 72.

13. Find the altitude and the area of an equilateral triangle whose side is 4 in.

14. Find the altitude and the area of an isosceles right triangle if the hypotenuse is 120.

B Exercises

1. A house with a gable roof is 45 ft. long, 32 ft. wide, 24 ft. to the roof, and 30 ft. to the ridgepole. Find the number of sq. ft. in the entire exterior surface.

2. The height of a lean-to roof is 8 ft.; the span is 15 ft. Find the area of the roof if the length is 21 ft. (see Fig. 458).

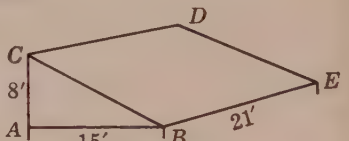


FIG. 458

3. A rectangle is 24 in. by 15 in. An equilateral triangle whose base is equal to the base of the rectangle stands upon the rectangle. Find the total height and area of the figure.

4. Solve Ex. 3 if the equilateral triangle had been an isosceles triangle with one leg 37 in.

5. The area of a rectangle is 120 sq. ft.; one side is 8 ft. Find the diagonal.

6. Upon the diagonal of a rectangle 11 ft. wide and 60 ft. long a triangle is constructed whose area equals the area of the rectangle. Find its altitude.

7. One side of a right triangle is 8 in. and the hypotenuse is 17 in. Find the area of an equilateral triangle constructed on the third side.

8. A square and an equilateral triangle have each a perimeter of 60 ft. Find the ratio of their areas.

9. The area of a rhombus is one-half the product of the diagonals.

Analysis:

To find area $ABCD$ (Fig. 459), find the area of $\triangle ABD$ and of $\triangle BDC$ and add (see p. 135, Ex. 3, and §307, p. 289).

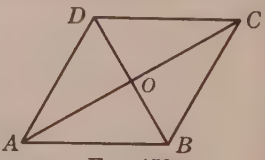


FIG. 459

10. The perimeter of a rhombus is 100 ft.; the shorter diagonal is 14 ft. Find the area. Use Ex. 9.

11. The area of a rhombus is 240 sq. ft.; the shorter diagonal is 16 ft. Find the longer diagonal and one side.

12. The parallel sides of an isosceles trapezoid are 30 ft. and 40 ft. respectively; the non-parallel sides are each 13 ft. Find the altitude and the area.

A Exercises

The figures in the exercises below are from various phases of industrial ornament.

1. Fig. 460 (a parquet floor design) is a square with the lines drawn symmetric with respect to both medians. If $AB = 12$ in. and $AX = 3$ in., find the area of the shaded portion.

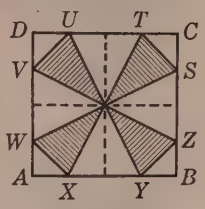


FIG. 460

2. In Fig. 461, $\angle A$ = 1 rt. $\angle$. $AB = AD$ and AC is the perpendicular bisector of DB. If $AB = 4$ in. and $AC = 8\frac{1}{4}$ in., find the area of $ABCD$ and the length of DC.

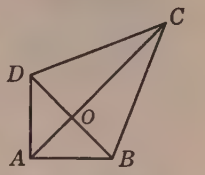


FIG. 461

3. Prove that the area of the kite-shaped figure in Fig. 461 is $\frac{1}{2} BD \cdot AC$. The method is similar to that for Ex. 9, p. 320.

4. The design shown in Fig. 462 is symmetric with regard to both diagonals of the square. If $AY = \frac{1}{4}AB$ and $OX = \frac{1}{3}OA$, find the area of $AYXZ$ and hence of the Maltese cross shown. $AB = 6$ in. (A parquet floor design.)

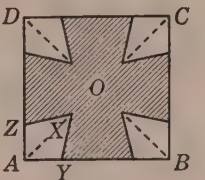
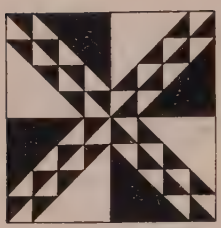


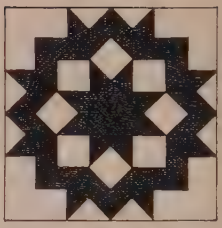
FIG. 462

5. The mid-points of the sides of a given square are joined in order. Find the area of the new square formed if one side of the given square is 6 in.

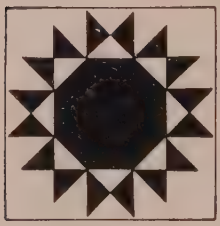


No. 1. Path through Woods

No. 2. Wedding Ring

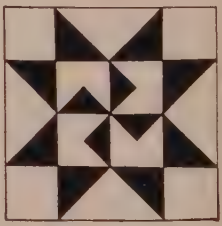


No. 3. Dutch Rose

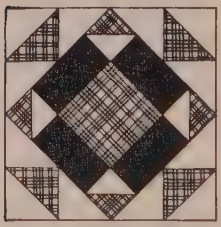


No. 4. Wheel of Fortune

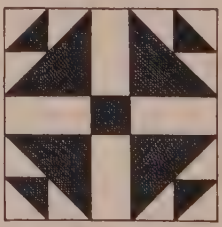
No. 5. Star of the East



No. 6. Octagonal Star

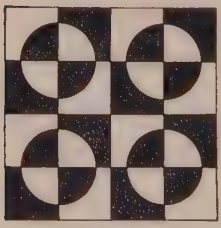


No. 7. Scotch Plaid

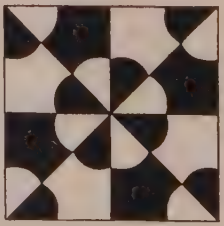


No. 8. Wild Goose Chase

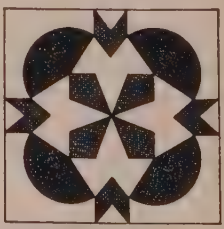
No. 9. Capital T



No. 10. Indiana Puzzle



No. 11. Dutch Windmill



No. 12. Farmer's Wife

FIG. 463. **Geometric design units used in old-fashioned quilts**

6. Fig. 464 represents a square with each side divided into 8 equal parts. What fraction of the given square is each of the irregular figures shown? If AB is 6 in., find the length of the sides of each figure.

NOTE: Modern standard tiles are made in certain definite sizes and shapes, all of which are derived directly from the six-inch square. Fig. 464 shows three such tiles.

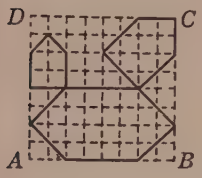


FIG. 464

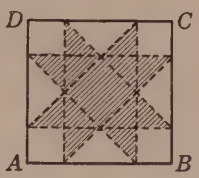


FIG. 465

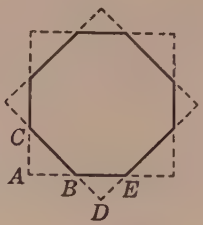


FIG. 466

7. Fig. 465 represents a square with each side divided into 4 equal parts and the points joined as indicated. If $AB = 12$ in., find the area of the star.

8. Fig. 466 shows a square with each side divided into three equal parts and the points joined as indicated. If each side of the original square is 12 in., find the area of the eight-pointed star and of the irregular octagon in the center.

9. Fig. 467 shows a square with AX and CY each one-fourth the diagonal AC. If the points are joined as indicated, prove that $XBYD$ is a rhombus and find its area. $AB = 6$ in.

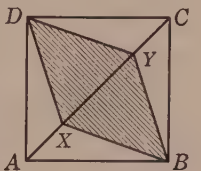


FIG. 467

10. Find the ratio of the area of the rhombus $XBYD$ in Fig. 467 to the area of the original square.

11. Could you find the area of any of the patterns shown in Fig. 463? What data would be necessary in each case if a side of each square is 12 in.?

A Exercises

NOTE. Many of the following exercises should be solved by algebraic equations. Use formulas, pp. 317, 318.

1. If the length of a rectangle is 3 ft. more than the width and the area is 70 sq. ft., find the dimensions.

2. If the length of a rectangle is two and one-half times the width and the area is 360, find the dimensions.

3. The length of a rectangle is 14 ft. more than its width. If its diagonal is 26 ft., find the dimensions and area.

4. Find the dimensions of a rectangle if its perimeter is 38 in. and its area 84 sq. in.

‡5. Construct a parallelogram whose sides are 5 cm. and 8 cm. and one of whose angles is 60°. Find its altitude and area.

‡6. Construct a parallelogram whose sides are 6 cm. and 9.4 cm. and one of whose angles is 30°. Find its altitude and area.

7. The base of a triangle is 3 ft. more than its altitude. Find the base and the altitude if the area is 90 sq. ft.

8. The base of a triangle is three times its altitude. Find the base and the altitude if the area is 336 sq. ft.

9. There are other methods of drawing construction lines in the trapezoid $ABCD$ (Fig. 468) so as to divide it into parts whose areas may be found and added. Prove Th. 74, p. 318, by means of Fig. 468 and suggest other possible figures.

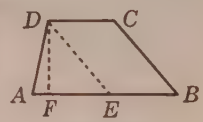


FIG. 468

10. Prove Th. 74, p. 318, by using Fig. 469. X is the mid-point of BC.

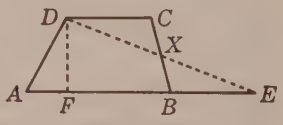


FIG. 469

11. The base and altitude of a triangle are 18 ft. and 24 ft. respectively. Ten ft. from the base a line is drawn parallel to the base. Find the area of the two parts into which the triangle is divided. Use an equation.

‡ Depends upon optional work.

12. *Prove that the area of an isosceles right triangle is* $\frac{b^2}{4}$ *if the hypotenuse is b.*

13. If the area of an isosceles right triangle is 120, find the hypotenuse and one leg.

14. *Prove that the area of an equilateral triangle is* $\frac{b^2}{4}\sqrt{3}$ *if the base is b.*

15. If the area of an equilateral triangle is 80, find one side.

16. Upon the altitude of an equilateral triangle as a side a second equilateral triangle is constructed. Find the area of the second triangle if a side of the first is 6 in.

17. If the parallel sides of a trapezoid are 8 in. and 20 in. and its altitude is 6 in., find the area of the two triangles formed by extending the non-parallel sides until they intersect.

18. The legs of a right triangle are 15 ft. and 20 ft. A perpendicular is drawn from the vertex of the right angle to the hypotenuse. Find the areas of the two triangles formed. Use an equation.

19. In Fig. 470, ABC is an isosceles right triangle. DE is parallel to BC. F is the mid-point of DE. FG and FH are parallel to AB and AC respectively. If $BC = 9$ in. and $AD = 4\frac{1}{2}$ in., find the areas of the various parts into which the figure is divided.

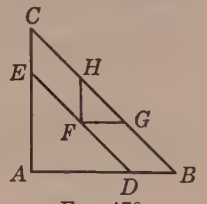


FIG. 470

20. The diagonals of two squares are 10 ft. and 15 ft. respectively. Find the diagonal of a square equivalent to their sum.

‡21. Two streets intersect at right angles. A third street cuts the other two at angles of 30° and 60°. The shortest side of the triangular park left is 200 feet. Find the area of the park at their intersection (Fig. 471).

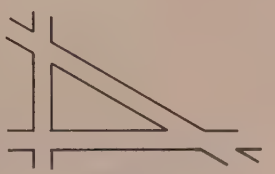


FIG. 471

‡Depends upon optional work.

329. (*Optional.*) **Measurement of irregular polygons.** Surveyors and others who use mathematics frequently find it necessary to find the areas of irregular polygons.

One method is to divide the figure into triangles and trapezoids and, after taking the proper measurements, find the areas of the various parts (see Ex. 1, below).

Sometimes the field to be surveyed is bounded on one side by a stream, the shore of a lake, or a curved road. In such cases a straight line is run near the curved boundary, and the inclosed figure is cut into trapezoidal shaped figures by offsets run perpendicular to the line, as shown in Fig. 475.

In such cases care should be taken that the curved boundaries of the figures are as nearly straight as possible.

The area of an irregular polygon may be found approximately by weighing. Cut the figure and also a square unit from the same sheet of paper or cardboard and weigh each. The areas have the same ratio as the weights. Archimedes used this method. Surveyors sometimes use it today.

Ex. 1. Compute the area of the field shown in Fig. 472 from the following data: $AC = 270$ ft., $h = 104$ ft., $k = 82$ ft.

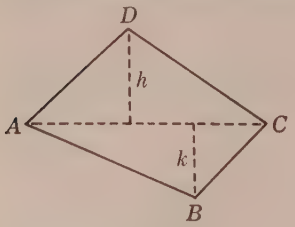


FIG. 472

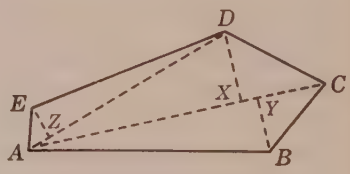


FIG. 473

Ex. 2. Find the area of the irregular field shown in Fig. 473 using the following data: $AC = 127$ ft., $AD = 97$ ft., $BY = 21$ ft., $DX = 32$ ft., $EZ = 13$ ft.

Ex. 3. Find the area of the gable shown in Fig. 474. $ABCD$ is an isosceles trapezoid and DCE an isosceles triangle. Use the dimensions given.

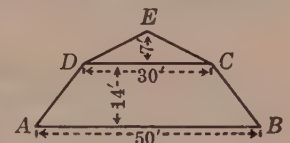


FIG. 474

Ex. 4. Find the area inclosed between the fence AF (Fig. 475) and the river from the following data:

$AB = 150$ ft. $BB' = 140$ ft.
$BC = 140$ ft. $CC' = 180$ft.
$CD = 160$ ft. $DD' = 95$ ft.
$DE = 250$ ft. $EE' = 120$ ft.
$EF = 100$ ft.

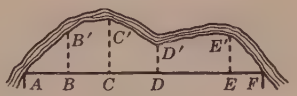


FIG. 475

Ex. 5. Find the area of the cross section of the girder shown in Fig. 476. The figure is symmetric with respect to both vertical and horizontal axes. $AYEF$ is a trapezoid with $AF \perp AY$. The total length is 5.2 in., $XY = .3$ in., $AF = .3$ in., $EY = .5$ in., and $AB = 3.8$ in.

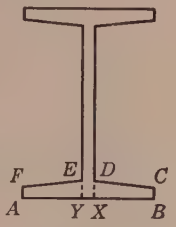


FIG. 476

Ex. 6. In Fig. 477, $XY = 50$, $YZ = 20$, $ZW = 55$, $AX = 63$, $DY = 120$, $BZ = 30$, $CW = 60$. Find the area of $ABCD$. AX, DY, BZ, and CW are each perpendicular to XW.

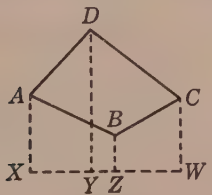


FIG. 477

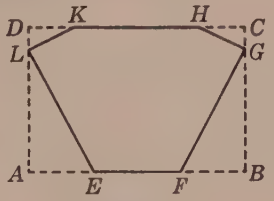


FIG. 478

Ex. 7. Find the area shown in Fig. 478. $ABCD$ is a rectangle. $DK = HC = 20$ ft., $DC = 100$ ft., $CG = DL = 10$ ft., $EF = 40$ ft., $EL = FG = 65$ ft., and $AE = FB$.

TOPIC 2. EQUAL AREAS

330. Th. 72, p. 316, and Th. 73, p. 317, give the following:

COR. I. **Parallelograms with equal bases and equal altitudes have equal areas.**

COR. II. **Triangles with equal bases and equal altitudes have equal areas.**

331. The axioms (§51, pp. 29 and 30) may be applied to equal areas; for example:

Ax. 1. If equal areas are added to equal areas, the results are equal areas.

Ax. 2. If equal areas are subtracted from equal areas, the results are equal areas.

In §318, p. 311, and §319, p. 312, you saw that two polygons cover the same extent of surface if they are made up of parts congruent in pairs. This general idea is now extended. You can prove that two polygons have equal areas by proving that they are sums, differences, equal multiples, or equal parts of polygons with equal areas. Such polygons have the same size but not the same shape.

C Exercises

1. In Fig. 479 draw the altitude of each triangle and find three triangles with equal areas. $EC \parallel AB$.

NOTE. The altitude of a triangle must be drawn from the vertex.

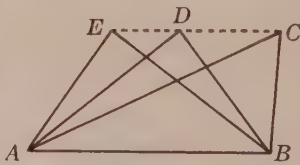


FIG. 479

2. Using any $\square ABCD$, construct two parallelograms that have the same area as $ABCD$, (*a*) using AB as the common base and (*b*) using AD as the common base.

3. Find elsewhere in the book illustrations of triangles or parallelograms with equal areas and tell why the areas are equal.

4. Construct a right triangle that shall have AB as one leg (Fig. 480) and its area equal to the area of $\triangle ABC$.

Suggestion. Show that the vertex of the triangle must lie on a line through $C \parallel AB$.

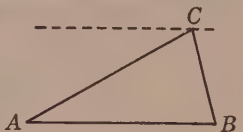


FIG. 480

5. Construct an isosceles triangle that shall have AB (Fig. 480) as base and the same area as $\triangle ABC$.

6. Given (Fig. 481) $\square ABCD$ with X any point in the diagonal AC. Prove area ABX = area ADX.

Suggestion. $\triangle ABX$ and $\triangle ADX$ must have equal bases and equal altitudes. Use AX as the common base.

To prove altitude DE = altitude BF, prove $\triangle ADE \cong \triangle BCF$.

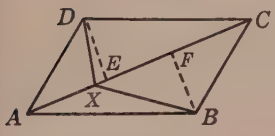


FIG. 481

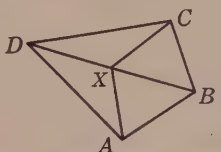


FIG. 482

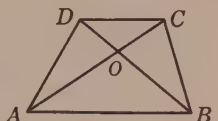


FIG. 483

7. Given (Fig. 482) the quadrilateral $ABCD$ with X the mid-point of the diagonal DB. Prove area $AXCD$ equals area $AXCB$.

Suggestion. Use Ax. 1, §331, p. 328.

8. Given (Fig. 483) trapezoid $ABCD$ with its diagonals. Prove area AOD = area BOC.

Suggestion. Use Ax. 2, §331, p. 328.

PROBLEM 17 (*Optional*)

332. To construct a triangle whose area shall be equal to the area of a given polygon.

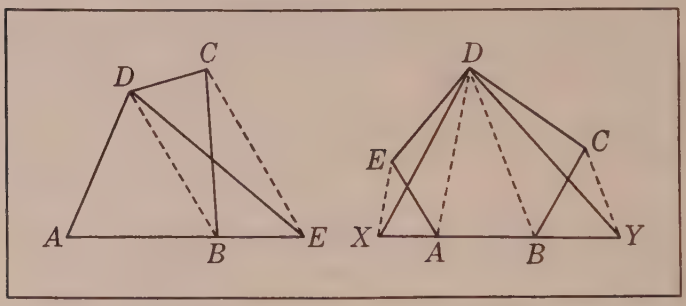


FIG. 484 FIG. 485

Given the polygon $ABCD$ etc.

To construct a triangle whose area shall be equal to the area of $ABCD$ etc.

Case A. When the polygon has four sides.

Analysis (Fig. 484):

1. Suppose the base of the triangle is to be in the line AB and its vertex at D, then $\triangle ADB$ will be part of the required triangle.
2. $\therefore$ it is necessary to construct a $\triangle$ with the same area as $\triangle DBC$ with DB for one side and another side on AB extended.

Directions: Draw DB; draw $CE \parallel DB$ from C. Join DE. ADE is the required triangle.

Outline of proof. Give the reasons.

1. Area $\triangle DBE$ = area $\triangle DBC$.
2. Area $\triangle ABD$ = area $\triangle ABD$.
3. $\therefore$ area $(\triangle DBE + \triangle ABD)$ = area $(\triangle DBC + \triangle ABD)$.

Case B. When the polygon has more than four sides (Fig. 485).

Directions in outline. Give details.

1. Draw DA and DB.
2. Construct $\triangle ADX$ with the same area as $\triangle ADE$. and $\triangle BDY$ with the same area as $\triangle BDC$.

Outline of proof. Give the reasons.

1. Area $\triangle ADX$ = area $\triangle ADE$.
2. Area $\triangle ABD$ = area $\triangle ABD$.
3. Area $\triangle BDY$ = area $\triangle BDC$.
4. $\therefore$ area$(\triangle ADX + \triangle ABD + \triangle BDY)$ = area$(\triangle ADE + \triangle ABD + \triangle BDC)$.

Exercise. (*Optional.*) Extend the method given in Case B above to a polygon of six sides.

B Exercises

1. Divide a triangle into three parts that have equal areas by segments drawn from the vertex to the base.

2. The line joining the mid-points of the parallel sides of a trapezoid divides it into two parts that have equal areas.

‡**3.** Given any quadrilateral $ABCD$. Construct a triangle that shall have the same area as the quadrilateral and that shall have:

a. Its base in the line AB and vertex at C.
b. Its base in the line BC and vertex at A.
c. Its base in the line BC and vertex at D.
d. Its base in the line AD and vertex at C.

4. If the mid-points of two sides of a triangle are joined to any point of the base, the quadrilateral formed has half the area of the triangle.

5. The triangle formed by joining the mid-point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side has half the area of the trapezoid.

‡ Depends upon optional work.

6. *The area of a triangle is one-half the product of the perimeter and the radius of the inscribed circle.*

7. The area of any polygon circumscribed about a circle is one-half the product of the perimeter and the radius of the inscribed circle.

8. Given $ABCD$, any quadrilateral, with E and F the mid-points of AB and CD respectively. Prove area $AECF$ equals area$(\triangle AFD + \triangle EBC)$ (Fig. 486).

Suggestion. Draw AC.

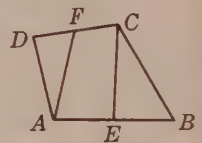


FIG. 486

9. Two triangles have equal areas if two sides of one are equal respectively to two sides of the other and the angles included by these sides are supplementary.

10. In Fig. 487, $ABCD$ is a square with its diagonals and medians. The semi-medians, OE, OF, OG, and OH, are bisected at X, Y, Z, and W, respectively. Prove that the area of the star formed is one-half the area of the square. Figs. 487 and 488 are from parquet floor designs.

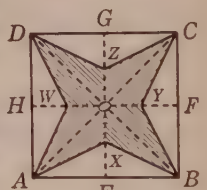


FIG. 487

11. In Fig. 488, $ABCD$ is a square with its diagonals and medians. OX, OY, OZ, and OW are each one-third of the corresponding semi-diagonal. Compare the area of the star with the area of the square $ABCD$.

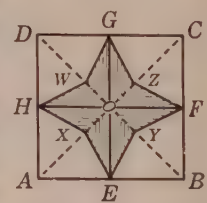


FIG. 488

12. If the mid-points of two adjacent sides of a parallelogram are joined, the triangle formed has one-eighth the area of the parallelogram.

13. A quadrilateral is bisected by one of its diagonals if the second diagonal is bisected by the first.

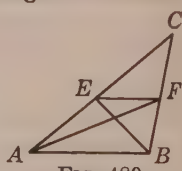


FIG. 489

14. Given $\triangle ABC$ with $EF \parallel AB$. Prove the area of $\triangle ACF$ equals the area of $\triangle BCE$ (Fig. 489).

15. Given the $\square ABCD$ with O any point within the parallelogram. Prove that the area of $(\triangle AOB + \triangle DOC)$ is equal to one-half the area of $\square ABCD$.

16. If, in Ex. 15, point O were without the $\square ABCD$, what would be the relation between the area of $\triangle AOB$ and $\triangle DOC$ and that of $\square ABCD$?

17. If through any point on the diagonal of a parallelogram segments are drawn parallel to the sides, the parallelograms on opposite sides of the diagonal have equal areas (see Fig. 490).

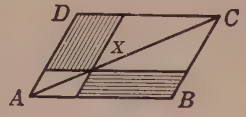


FIG. 490

18. Given $\triangle ABC$ with X an arbitrary point in the median CO. Prove that the area of $\triangle ACX$ equals the area of $\triangle BCX$.

19. Given (Fig. 491) $\triangle ADC$ and DBE with $AD = DB$, $DE = EC$, $\angle ADC + \angle BDE =$ 2 rt. $\angle$. Prove area $DBE = \frac{1}{2}$ area ADC.

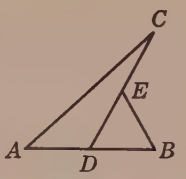


FIG. 491

20. The three medians of a triangle divide it into six triangles that have equal areas.

21. An isosceles right triangle has one-fourth the area of the square constructed on the hypotenuse.

22. The figure formed by joining the mid-points of the sides of a rhombus has one-half the area of the rhombus.

23. If one base of a trapezoid is twice the other, either diagonal divides the trapezoid into two parts such that the area of one is double the area of the other.

24. ABC is an isosceles right triangle. G is the mid-point of the hypotenuse AB. GF and GE are parallel to AC and BC respectively. Prove area $ECFG$ equal to $\frac{1}{2}$ area ABC. (See §163, p. 143.)

25. Would Ex. 24 be true if (*a*) ABC is any right triangle? (*b*) ABC is any triangle and AB is any side? Give proof.

THEOREM 75 (*Optional*)

333. The area of the square constructed on the hypotenuse of a right triangle is equal to the sum of the areas of the squares constructed on the other two sides.

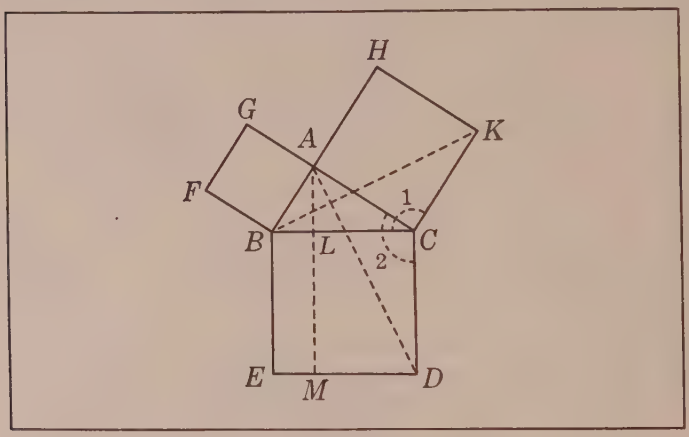


FIG. 492

Hypothesis: ABC is a rt. $\triangle$ with $\angle A$ a rt. $\angle$. BD is the square on the hypotenuse BC. AF and AK are the squares on the other two sides.

Conclusion: Area BD = area AF + area AK.

Analysis and construction:

1. Area BD may be proved equal to area AF + area AK by dividing BD into two parts whose areas equal area AF and area AK respectively.
2. One way of dividing BD into the two required parts is by drawing a line from $A \perp BC$ and proving area $BEML$ = area AF and area $CDML$ = area AK.
3. To prove area $CDML$ = area AK, compare them with congruent triangles.

4. To obtain the triangles, join AD and BK. Prove

a. $\triangle ACD \cong \triangle BCK$.

b. 2 area ACD = area $CDML$.

c. 2 area BCK = area AK.

Outline of proof. Give all reasons.

1. *a.* $\angle 1 = \angle 2$.

b. $CK = CA$.

c. $CB = CD$.

d. $\therefore \triangle BCK \cong \triangle ACD$.

2. *a.* HAB is a straight line.

b. $\therefore AC$ is equal to the altitude of $\triangle BCK$.

c. Area $BCK = \frac{1}{2}CK \cdot AC$.

d. Area $AK = CK \cdot AC$.

e. $\therefore$ area AK = 2 area BCK.

3. *a.* LC is equal to the altitude of $\triangle ACD$.

b. Area $ACD = \frac{1}{2}LC \cdot CD$.

c. Area $CDML = LC \cdot CD$.

d. $\therefore$ area $CDML$ = 2 area ACD.

4. *a.* $\therefore$ area $CDML$ = area AK.

b. Similarly, area $BEML$ = area AF.
(Give details.)

c. $\therefore$ area $(CDML + BEML)$ = area AK + area AF,

d. or area BD = area AK + area AF.

334. Cor. *(Optional.)* **The area of the square on one leg of a right triangle is equal to the area of the square on the hypotenuse minus the area of the square on the other leg, or**

area AF = area BD − area AK

PROBLEM 18 (*Optional*)

335. To construct a square whose area is equal to the sum of the areas of two given squares.

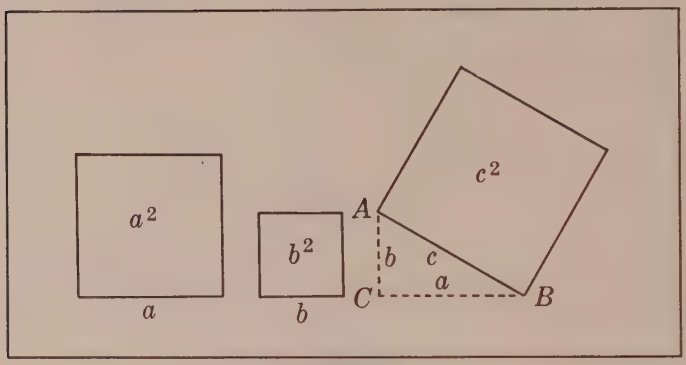


FIG. 493

Given the squares a^2 and b^2 whose sides are segments a and b respectively.

To construct a third square c^2 such that $c^2 = a^2 + b^2$.

Directions: Construct a right triangle with a and b as legs. The square on the hypotenuse is c^2. Why?

Ex. 1. Construct a square whose area is equal to the difference between the areas of two given squares.

Ex. 2. Construct a square whose area is equal to the sum of the areas of three given squares.

A Exercises—The Pythagorean Theorem

NOTE. While Th. 75 is called the Pythagorean Theorem (see note, p. 290), the proof given in §333, p. 334, was probably invented by Euclid (see note, p. 78). The first two exercises given on the following page are variations of Euclid's proof.

1. In the details for the proof of Th. 75, as given on page 335, prove that $\triangle ACD$ and BCK have equal bases and equal altitudes instead of proving them congruent (Fig. 494).

Analysis:

I. To prove $\triangle ACD$ and $\triangle BCK$ have equal bases and equal altitudes, use CD and BC as bases and prove the altitudes equal.

II. $\therefore$ draw $\perp$s from A and K to CD and BC extended respectively and prove $\triangle ACY \cong \triangle KCX$.

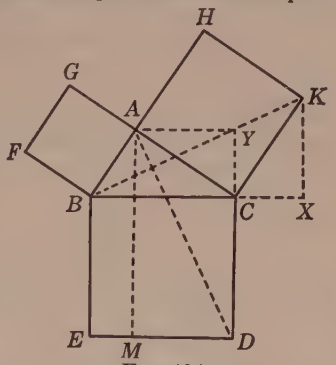


FIG. 494

2. Give the details for the proof for Th. 75 if (*a*) square BD is on the opposite side of BC from that shown in Fig. 494; (*b*) square AK is on the opposite side of AC from that shown in the figure; (*c*) both squares BD and AK are on the opposite sides of BC and AC from those shown in the figure.

3. Use Fig. 495 to prove Th. 75. ABC is the given triangle. Extend CB, making $BD = b$. Draw $ED \perp DC$ and equal to a. Draw BE and AE. Prove ABE an isosceles right triangle. Find the area of trapezoid $ACDE$ and set it equal to the sum of the areas of the three triangles and simplify the result. (This is President Garfield's proof).

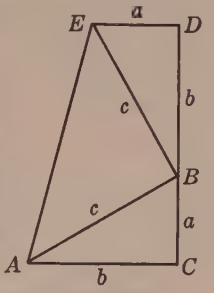


FIG. 495

4. A Hindu mathematician of the 12th century used a figure like Fig. 496 to verify Th. 75. ABC is the given triangle and $ABDE$ the square constructed on AB. How is the figure constructed? If $b = 2a$, cut out the parts and rearrange them so as to show the sum of the squares on b and a.

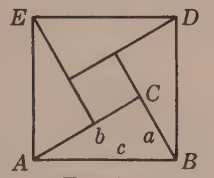


FIG. 496

A Exercises — Constructions Based on Algebraic Analysis

1. On a given segment as base construct a rectangle that shall have the same area as a given parallelogram.

Analysis (Fig. 497):

I. Let b represent the base and h the altitude of the given parallelogram. Then its area is hb.

II. Let k represent the base and x the unknown altitude of the rectangle. Then its area is kx.

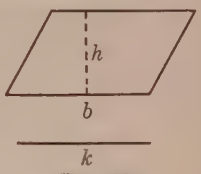


FIG. 497

III. To construct the rectangle, find x.

IV. Since the areas are to be equal, $kx = hb$.

V. $\therefore$ find x so that $kx = hb$, that is, so that $\frac{k}{b} = \frac{h}{x}$ (see §284, p. 265).

2. Construct a square that shall have the same area as a given parallelogram.

Suggestion. Make an analysis similar to that for Ex. 1. Let x be the side of the unknown square. Show that it is necessary to construct x so that $x^2 = hb$, that is, so that $\frac{h}{x} = \frac{x}{b}$ (see §305, p. 288).

3. On a given segment as base construct an isosceles triangle that shall have the same area as a given parallelogram.

Suggestion. Make an analysis similar to that for Ex. 1. Let x be the unknown altitude of the required triangle. Show that it is necessary to construct x so that $\frac{1}{2}kx = hb$, that is, so that $\frac{\frac{1}{2}k}{b} = \frac{h}{x}$.

On a given segment as base, construct:

4. A rectangle equal in area to a given triangle.

5. An isosceles triangle equal in area to a given square.

6. An isosceles triangle equal in area to any triangle.

Construct a square whose area shall be equal to:

7. Any given triangle.

8. Three times the area of a given square.

9. One-half the area of a given square.

10. Construct on a given segment as base an isosceles triangle whose area shall be three-fourths the area of a given square.

11. *Construct a square that has the same area as a given polygon.*

12. Construct on a given segment as hypotenuse a right triangle whose area shall be equal to the area of a given square.

13. Given a triangular field ABC. Show how to divide it into two parts that shall have equal areas, by running a fence from point E on one side.

Analysis and construction (Fig. 498):

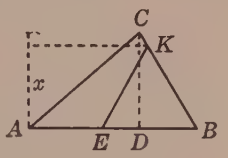


FIG. 498

I. This is the same as constructing on EB as base a triangle whose area shall be $\frac{1}{2}$ area ABC.

II. Let x be the unknown altitude of the new triangle.

III. $\therefore$ find x so that $\frac{1}{2}x \cdot EB = \frac{1}{2} \cdot \frac{1}{2}CD \cdot AB$.

IV. $\therefore$ find x so that $x \cdot EB = \frac{1}{2}CD \cdot AB$.

V. $\therefore$ find x so that $\frac{EB}{AB} = \frac{\frac{1}{2}CD}{x}$.

The remainder of the directions is suggested by the figure.

Discussion:

Let the pupil make the drawing and note the position of K when (*a*) E is the mid-point of AB; (*b*) $EB < \frac{1}{2}AB$; (*c*) $EB > \frac{1}{2}AB$; (*d*) E is very near A. Can EK be $\parallel AC$? If so, when?

Survey Test

1. Show by a drawing that if you add congruent polygons to congruent polygons, (*a*) you might get congruent polygons as a result; (*b*) you might not get congruent polygons as a result.

2. Answer question 1 for the subtraction instead of the addition of congruent polygons.

3. (*a*) Define the area of a polygon. (*b*) What is meant by measuring the surface of a polygon?

Complete each of the 4 following statements:

4. The area of a rectangle is . . . . . .

5. The area of a parallelogram is . . . . . .

6. The area of a triangle is . . . . . .

7. The area of a trapezoid is . . . . . .

8. (*a*) State each of the above four theorems as a formula. (*b*) What do each of the letters in the formulas stand for?

9. Give the first step in the analysis for each of the theorems quoted in questions 7, 6, and 5 above, in the order here stated.

10. How would you prove that (*a*) two parallelograms have the same area? (*b*) two triangles? (*c*) a triangle and a parallelogram?

11. (*a*) How would you prove that any two polygons have the same area. (*b*) What special case of this have you had?

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*	*Question*	*Page*
1	311	7	318
3	312	8	317, 318
4	313	9	318, 317, 316
5	316	10	328
6	317	11	328

UNIT VIII

SIMILAR POLYGONS

336. Similar figures. On page 271 you began the study of similar figures—that is, of figures that have the same shape but not the same size. Such figures abound in the world about you. The layout of the floors of a house which is under construction is similar to the architect's floor plan (page 8). Indeed, a great building like the Empire State Building in New York could not be built without the use of the architect's blueprints. The map of the streets of a city is similar to the horizontal layout of the city. The surveyor makes a plan of a farm that is similar to the horizontal lines bounding the farm. An enlargement of a photograph is similar to the original; the picture thrown on a screen by a motion-picture machine is similar to the picture on the film. The illustration on page 272 provides a good example of similar figures. Illustrated sales catalogs contain pictures that are similar to the articles illustrated. In all such cases corresponding angles are equal and corresponding segments have equal ratios. Many necklaces are made of graduated stones of different sizes cut in the same shape. Leaves approximately congruent or similar are often found on the same plant (page 342). Name other illustrations of the occurrence of similar figures. Do you know of any plant the leaves of which are not similar?

337. When are two **polygons similar** in geometry? What is meant by the **ratio of similitude** of two similar polygons? (See page 271). For convenience, corresponding sides will be lettered alike, as AB and $A'B'$, BC and $B'C'$, etc.

338. Review Th. 65, p. 273, and its proof. Use Th. 65 to prove the following:

Ex. 1. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

Ex. 2. Two isosceles triangles are similar if an angle of one is equal to a corresponding angle of the other (two cases).

Ex. 3. Perpendiculars from any two points on one side of an angle to the other side cut off similar triangles.

Ex. 4. A line parallel to the base of a triangle and intersecting two sides cuts off a triangle similar to the given triangle.

Ex. 5. Prove Ex. 4 if the line parallel to one side cuts the other two sides extended (*a*) through the vertex; (*b*) beyond the base.

Ex. 6. If you have at home two serving trays that set nicely one into the other, measure the dimensions of each and find the ratios of corresponding sides. Are the trays similar?

FIG. 499. **Japanese morning glory leaves, showing similarity in nature**

NOTE. In geometry you have studied congruent figures, similar figures, and figures that have the same area. In popular language, figures that have the same area are said to have the same size, similar figures the same shape, and congruent figures both the same size and shape. Figures that have the same shape but not the same size abound in nature and art. Are figures that have the same shape necessarily similar?

THEOREM 76*

339. Two triangles are similar if an angle of one is equal to an angle of the other and the ratios of the including sides are equal.

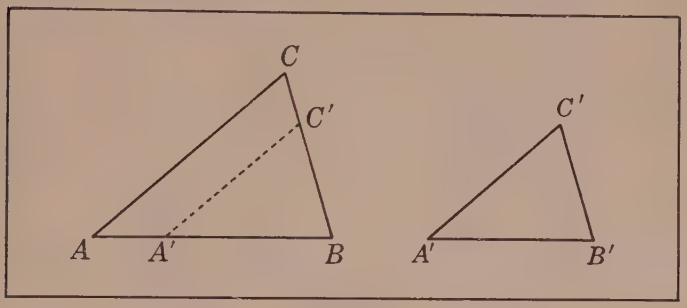


FIG. 500

Hypothesis: In $\triangle ABC$ and $\triangle A'B'C'$, $\angle B = \angle B'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.

Conclusion: $\triangle ABC \sim \triangle A'B'C'$.

Analysis and construction:

1. To prove $\triangle ABC \sim \triangle A'B'C'$, prove $\angle A = \angle A'$. (See §294, p. 274.)
2. To prove $\angle A = \angle A'$, place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle B'$ falls on $\angle B$, with $B'A'$ along BA and $B'C'$ along BC, and prove $A'C' \parallel AC$.
3. To prove $A'C' \parallel AC$, prove $\frac{BA}{BA'} = \frac{BC}{BC'}$.

Ex. 1. Construct two similar triangles with the ratio of similitude two-thirds.

Ex. 2. Given $\triangle ABC$ any triangle and $A'B'$ of $\triangle A'B'C'$. Construct $\triangle A'B'C' \sim \triangle ABC$, basing the construction (*a*) on §294, p. 274, (*b*) on Th. 76.

THEOREM 77* (*Optional*)

340. Two triangles are similar if the corresponding sides have equal ratios.

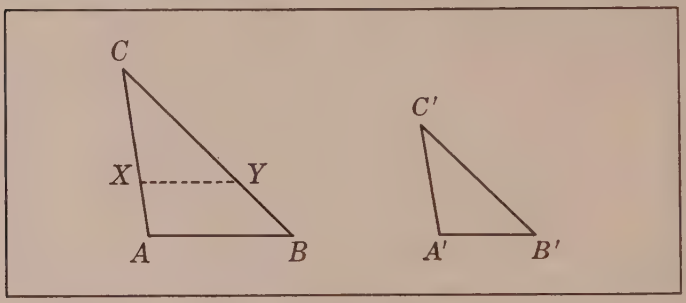


FIG. 501

Hypothesis: In $\triangle ABC$ and $\triangle A'B'C'$, $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CA}{C'A'}$.

Conclusion: $\triangle ABC \sim \triangle A'B'C'$.

Analysis and construction:

1. To prove $\triangle ABC \sim \triangle A'B'C'$, prove $\triangle ABC$ similar to a third triangle that is congruent to $\triangle A'B'C'$.
2. To obtain the new triangle, lay off CX on CA equal to $C'A'$, CY on CB equal to $C'B'$, and draw XY.
3. The new triangle is $\triangle CXY$. Prove $\triangle CXY$ similar to $\triangle ABC$ and congruent to $\triangle A'B'C'$.
4. To prove $\triangle CXY \cong \triangle A'B'C'$, prove $XY = A'B'$.
5. " " $XY = A'B'$, show that $\frac{CA}{C'A'}=\frac{AB}{A'B'}$ and $\frac{CA}{CX}=\frac{AB}{XY}$.
6. To prove $\frac{CA}{CX}=\frac{AB}{XY}$, prove $\triangle CXY \sim \triangle ABC$.

Proof:

	STATEMENTS	REASONS
1. *a.*	$\frac{CA}{C'A'}=\frac{CB}{C'B'}$.	1. *a.* Why?
b.	$\therefore \frac{CA}{CX}=\frac{CB}{CY}$.	*b.* Why?
c.	$\angle C=\angle C$.	*c.* Why?
d.	$\therefore \triangle CAB \sim \triangle CXY$.	*d.* Th. 76, p. 343.
2. *a.*	$\therefore \frac{CA}{CX}=\frac{AB}{XY}$.	2. *a.* Definition of similar triangles.
b.	$\frac{CA}{C'A'}=\frac{AB}{A'B'}$.	*b.* Given.
c.	$\therefore XY=A'B'$.	*c.* §264, p. 255.
3. *a.*	$XY=A'B'$.	
b.	$CX=C'A'$.	
c.	$CY=C'B'$.	
d.	$\therefore \triangle CXY \cong \triangle A'B'C'$.	
4.	$\therefore \triangle ABC \sim \triangle A'B'C'$.	4. It is similar to a triangle that is congruent to $\triangle A'B'C'$.

C Exercises

1. Two triangles similar to a third are similar.

2. Two right triangles are similar if their legs have equal ratios.

3. If two triangles are similar, they can be divided into pairs of similar triangles by corresponding medians.

4. Prove Ex. 3 for the bisectors of corresponding angles.

5. Prove Ex. 4, §338, p. 342, by Th. 76, p. 343.

6. Construct $\triangle A'B'C'$ similar to $\triangle ABC$ if the sum of two sides $A'B'$ and $B'C'$ is given (see Prob. 14, p. 263).

7. If one of two similar triangles is isosceles, the other is also.

THEOREM 78

341. Two polygons are similar if diagonals drawn from two corresponding vertices divide the polygons into the same number of triangles similar each to each and similarly placed.

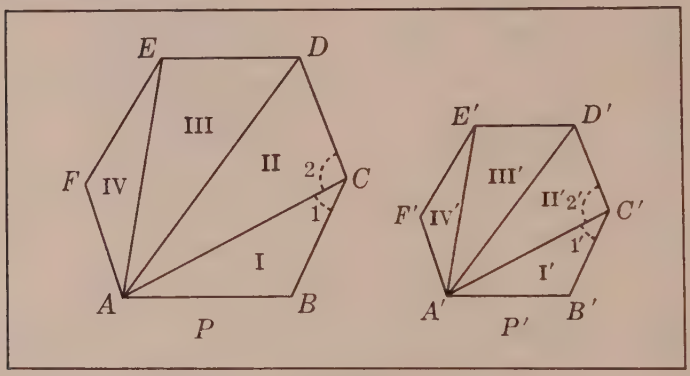


FIG. 502

Hypothesis: Polygons P and P' are divided into triangles by the diagonals drawn from the corresponding vertices A and A', so that $\triangle \text{I} \sim \triangle \text{I}'$, $\triangle \text{II} \sim \triangle \text{II}'$, etc.

Conclusion: Polygon $P \sim$ polygon P'.

Analysis:

1. To prove $P \sim P'$, prove (*a*) $\angle B = \angle B'$, $\angle C = \angle C'$, etc., (*b*) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \text{etc.}$
2. $\angle B = \angle B'$ since $\triangle \text{I} \sim \triangle \text{I}'$.
3. To prove $\angle C = \angle C'$, prove $\angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$.
4. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ since $\triangle \text{I} \sim \triangle \text{I}'$.
5. To prove $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, prove each ratio equal to $\dfrac{AC}{A'C'}$.

THEOREM 79*

342. If two polygons are similar, diagonals drawn from two corresponding vertices divide the polygons into the same number of triangles similar each to each and similarly placed.

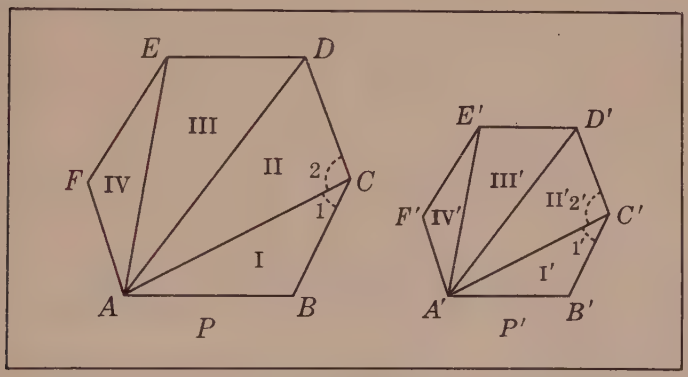


FIG. 503

Hypothesis: Polygon $P \sim$ polygon P'. Diagonals are drawn from the corresponding vertices A and A' dividing the polygons into triangles similarly placed.

Conclusion: $\triangle \mathrm{I} \sim \triangle \mathrm{I}'$, $\triangle \mathrm{II} \sim \triangle \mathrm{II}'$, etc.

Analysis:

1. *a*. To prove $\triangle \mathrm{II} \sim \triangle \mathrm{II}'$, prove $\angle 2 = \angle 2'$ and $\frac{AC}{A'C'} = \frac{CD}{C'D'}$.

   b. To prove $\angle 2 = \angle 2'$, prove $\angle C = \angle C'$ and $\angle 1 = \angle 1'$.

   c. " " $\frac{AC}{A'C'} = \frac{CD}{C'D'}$, prove each equal to $\frac{BC}{B'C'}$.

2. " " $\triangle \mathrm{I} \sim \triangle \mathrm{I}'$, prove $\angle B = \angle B'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.

For 2 use $P \sim P'$. For 1 use $P \sim P'$ and $\triangle \mathrm{I} \sim \triangle \mathrm{I}'$.

Problem 19

343. Upon a given segment corresponding to a given side of a given polygon, to construct a polygon similar to the given polygon.

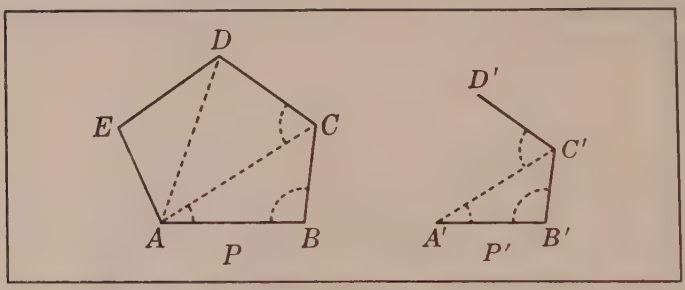


Fig. 504

Given polygon P and segment $A'B'$ corresponding to side AB in P.

To construct on $A'B'$ a polygon similar to P.

General plan: Base the construction on Th. 78, p. 346.

Directions:

1. Draw the diagonals of P from vertex A.
2. On $A'B'$ construct $\triangle A'B'C' \sim \triangle ABC$ (see p. 343, Ex. 2).
3. On $A'C'$ corresponding to AC construct $\triangle A'C'D' \sim \triangle ACD$.
4. Proceed in the same manner until P' is complete.

Proof: Why is $P' \sim P$?

Ex. 1. Two similar polygons are congruent if any two corresponding sides are equal.

Ex. 2. If two polygons are similar to a third, they are similar to each other.

Ex. 3. (*Optional.*) Construct P', Fig. 504, basing your construction on the definition of similar figures rather than on Th. 78, p. 346 (see Prob. 15, p. 265).

THEOREM 80

344. The ratio of corresponding altitudes of two similar triangles equals the ratio of any two corresponding sides.

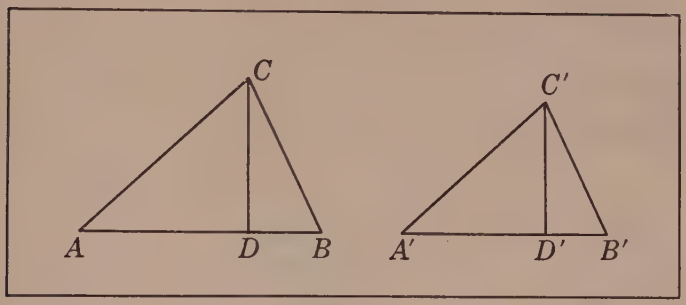


FIG. 505

Hypothesis: $\triangle ABC \sim \triangle A'B'C'$. CD and $C'D'$ are corresponding altitudes on the bases AB and $A'B'$ respectively.

Conclusion: $\dfrac{CD}{C'D'} = \dfrac{BC}{B'C'} = \dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.

Analysis:

1. To prove $\dfrac{CD}{C'D'} = \dfrac{BC}{B'C'}$, prove $\triangle CBD \sim \triangle B'C'D'$.

2. " " $\dfrac{CB}{C'B'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$, use $\triangle ABC \sim \triangle A'B'C'$ (given).

3. Complete by use of Ax. 5, p. 30.

Exercise. In two similar triangles the ratio of the following segments equals the ratio of similitude of the triangles:

a. Bisectors of corresponding angles;

b. Corresponding medians;

c. The radii of (1) the inscribed circles, (2) the circumscribed circles.

THEOREM 81

345. In a series of equal ratios, the ratio of the sum of the numerators to the sum of the denominators is equal to the ratio of any numerator to its denominator.

Hypothesis: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\text{etc.}$

Conclusion: $\frac{a+c+e+g+\text{etc.}}{b+d+f+h+\text{etc.}}=\frac{a}{b}$.

Outline of proof. Give the reasons.

1. $\frac{a}{b}=\frac{a}{b}, \frac{c}{d}=\frac{a}{b}, \frac{e}{f}=\frac{a}{b}, \frac{g}{h}=\frac{a}{b}$, etc.

2. $\therefore ab=ab.$
   $cb=ad.$
   $eb=af.$
   $gb=ah.$
   . . . . . .

3. $\therefore b(a+c+e+g+\text{etc.})=a(b+d+f+h+\text{etc.})$.
   (Addition Theorem, §307, p. 289.)

4. $\therefore \frac{a+c+e+g+\text{etc.}}{b+d+f+h+\text{etc.}}=\frac{a}{b}$. (§263, p. 255.)

Ex. 1. Verify Th. 81 by applying it to some numerical series, as

$$\frac{1}{2}=\frac{4}{8}=\frac{5}{10}=\frac{7}{14}$$

Ex. 2. If two polygons have the sides in equal ratios thus: $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$, how would Th. 81 apply?

Ex. 3. Are the polygons referred to in Ex. 2 similar? Why?

346. COR. **The ratio of the perimeters of two similar polygons equals the ratio of any two corresponding sides.**

Exercise. Two corresponding sides of two similar polygons are 20 and 25 respectively. If the perimeter of the first is 150, what is the perimeter of the second?

THEOREM 82*

347. The areas of two similar triangles have the same ratio as the squares of any two corresponding sides.

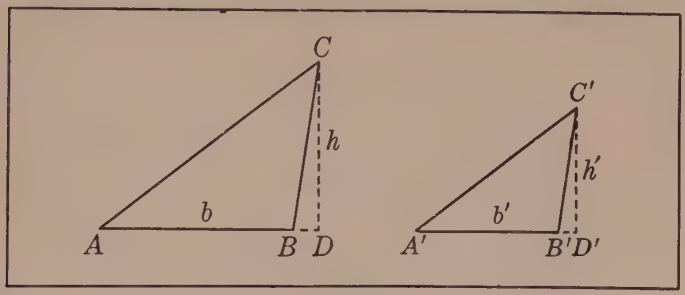


FIG. 506

Hypothesis: $\triangle ABC \sim \triangle A'B'C'$, b and b' are corresponding sides, and h and h' the altitudes on b and b' respectively.

Conclusion: $\dfrac{\text{Area } ABC}{\text{Area } A'B'C'} = \dfrac{b^2}{b'^2}$.

Analysis: To prove $\dfrac{\text{area } ABC}{\text{area } A'B'C'} = \dfrac{b^2}{b'^2}$, prove each ratio equal to a third.

Outline of proof. Give all reasons.

1. *a.* Area $ABC = \frac{1}{2}hb$ and area $A'B'C' = \frac{1}{2}h'b'$.

   b. $\dfrac{\text{Area } ABC}{\text{Area } A'B'C'} = \dfrac{hb}{h'b'} = \dfrac{h}{h'} \cdot \dfrac{b}{b'}$.

2. *a.* $\dfrac{h}{h'} = \dfrac{b}{b'}$.

   b. $\therefore \dfrac{hb}{h'b'} = \dfrac{b^2}{b'^2}$.

3. $\therefore \dfrac{\text{area } ABC}{\text{area } A'B'C'} = \dfrac{b^2}{b'^2}$.

THEOREM 83*

348. The areas of two similar polygons have the same ratio as the squares of two corresponding sides.

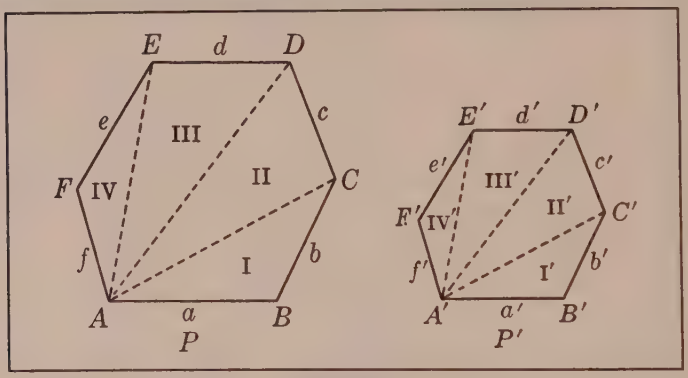


FIG. 507

Hypothesis: Polygon $P \sim$ polygon P', a and a' are corresponding sides.

Conclusion: $\dfrac{\text{Area } P}{\text{Area } P'} = \dfrac{a^2}{a'^2}$.

Analysis and construction:

1. To prove $\dfrac{\text{area } P}{\text{area } P'} = \dfrac{a^2}{a'^2}$, break up the polygon into triangles that are similar in pairs and prove $\dfrac{\triangle \text{I}}{\triangle \text{I}'} = \dfrac{\triangle \text{II}}{\triangle \text{II}'} = \dfrac{\triangle \text{III}}{\triangle \text{III}'} = \text{etc.} = \dfrac{a^2}{a'^2}$.

2. $\therefore$ draw diagonals from corresponding vertices.

Outline of proof. Give all reasons.

1. *a*. $\triangle \text{I} \sim \triangle \text{I}'$.

   b. $\therefore \dfrac{\triangle \text{I}}{\triangle \text{I}'} = \dfrac{a^2}{a'^2}$.

c. Similarly $\frac{\triangle II}{\triangle II'} = \frac{c^2}{c'^2}$.

d. $\frac{c}{c'} = \frac{a}{a'}$.

e. $\therefore \frac{c^2}{c'^2} = \frac{a^2}{a'^2}$.

f. $\therefore \frac{\triangle I}{\triangle I'} = \frac{\triangle II}{\triangle II'}$.

g. Similarly $\frac{\triangle II}{\triangle II'} = \frac{\triangle III}{\triangle III'}$.

h. $\frac{\triangle I}{\triangle I'} = \frac{\triangle II}{\triangle II'} = \frac{\triangle III}{\triangle III'} = \text{etc.}$

2. *a.* $\therefore \frac{\triangle I + \triangle II + \triangle III + \text{etc.}}{\triangle I' + \triangle II' + \triangle III' + \text{etc.}} = \frac{\triangle I}{\triangle I'}$.

b. But $\frac{\triangle I}{\triangle I'} = \frac{a^2}{a'^2}$.

c. $\therefore \frac{\triangle I + \triangle II + \triangle III + \text{etc.}}{\triangle I' + \triangle II' + \triangle III' + \text{etc.}} = \frac{a^2}{a'^2} = \frac{P}{P'}$.

C Exercises

1. The sides of a polygon are 10, 15, 9, and 22. Find the sides of a similar polygon if its perimeter is 140.

2. In two similar triangles let A and A' represent the areas, b and b' the bases, and h and h' the altitudes. Find

a. A', if $A = 45$ and $h = \frac{1}{2}h'$.

b. h', if $h = 16$ and $A = \frac{1}{3}A'$.

c. b', if $b = 10$ and $A' = 3A$.

3. The perimeters of two similar triangles have the same ratio as the altitudes.

4. The perimeters of two similar triangles are 21 and 36. The altitude of the first triangle is 8. Find the altitude of the second triangle.

5. The bases of two similar triangles are 24 and 42 respectively. If the perimeter of the first is 84, find the perimeter of the second.

6. Corresponding diagonals of two similar polygons have the same ratio as the perimeters.

7. The perimeters of two similar polygons are 144 and 216 respectively. If a diagonal of the first is 32, find the corresponding diagonal of the second.

8. If the perimeter of an equilateral triangle is 66, find the sides of a similar triangle with half the altitude.

9. If the area of one polygon is twice that of a similar polygon, what is the ratio of the corresponding sides?

10. The area of one polygon is three times the area of a similar polygon. If the corresponding side of one is 15, find the side of the other.

11. Two corresponding sides of two similar polygons are 5 and 9 respectively. If the area of the first is 30, find the area of the second.

12. The areas of two similar polygons are 225 and 400 respectively. If a side of the first is 30, find the corresponding side of the second.

13. The areas of two similar polygons have the same ratio as the squares of the perimeters or the squares of corresponding diagonals.

B Exercises

1. The equal sides of an isosceles triangle are 13, and the base is 10. Find the sides of a similar triangle whose altitude is 4.

2. The altitude of an isosceles triangle is 36, and the base is 96. Find the sides of a similar triangle whose altitude is 54.

3. The sum of the areas of two similar triangles is 78 sq. ft. Two corresponding sides are 10 ft. and 15 ft. Find the areas and altitudes.

4. The altitude of an isosceles triangle is 48 and one leg is 50. Find the base of a similar triangle if one leg is 21.

5. The sum of the areas of two similar polygons is 910. The ratio of similitude is seven-ninths. Find their areas.

6. On one side and on the altitude of an equilateral triangle as bases construct similar triangles. What is the ratio of their areas?

7. On one side and on the diagonal of a square construct equilateral triangles. What is the ratio of their areas?

8. Construct a triangle similar to a given triangle whose area shall be one-fourth the area of the given triangle; one-half the area; one-ninth the area; one-third the area.

9. What is the area in acres of a portion of a map that covers $4\frac{1}{2}$ sq. in.? The scale of the map is 1 inch = 1 mile.

Survey Test

1. (*a*) When are two polygons similar? (*b*) Define the ratio of similitude.

2. If you wish to prove two triangles similar, what methods might you use? Quote the theorems in full.

3. Do the theorems that might be used to prove triangles similar remind you in any way of the theorems that might be used to prove triangles congruent? Quote the theorem for similar triangles and with it the theorem for congruent triangles.

4. What theorem concerning congruent triangles has no corresponding theorem concerning similar triangles?

5. What theorem concerning similar triangles has no corresponding theorem concerning congruent triangles?

6. If two polygons are to be proved similar, what two conditions must be fulfilled? For what special case is this not true?

7. If you wish to prove two polygons of more than three sides similar, what methods might you use? Quote the authorities in full.

8. Explain how to construct, upon a given segment corresponding to a given side of a given polygon, a polygon similar to the given polygon.

9. What theorem concerning equal ratios is proved in this chapter? What purpose does it serve?

10. Name the properties of similar figures.

11. What property of similar polygons is proved directly from the same property of similar triangles?

12. Write the equation connecting b, b', h, and h' where b and b' are corresponding sides and h and h' corresponding altitudes of two similar triangles.

13. Write the equation connecting b, b', A, and A' where b and b' are corresponding sides and A and A' the areas of two similar polygons.

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	271	6	271	10	357
2	273, 343, 344	7	271, 346	11	352
4	49	8	348	12	349
5	273	9	350	13	352

349. SUMMARY III

Similar Figures, Ratio, Proportion, and Areas of Polygons

SIMILAR FIGURES

Two triangles are similar if:

1. They are mutually equiangular ..... Th. 65, p. 273
2. An angle of one equals an angle of the other and the sides including the angles are proportional ..... Th. 76, p. 343

‡3. Corresponding sides are proportional ..... Th. 77, p. 343

Two polygons are similar if:

1. The angles of one are equal respectively to the angles of the other and the corresponding sides are proportional ..... §291, p. 271
2. Diagonals from corresponding vertices divide the polygons into triangles that are similar and similarly placed ..... Th. 78, p. 346

Properties of similar figures:

1. The corresponding angles are equal ..... §291, p. 271
2. Corresponding sides have equal ratios ..... §291, p. 271
3. Diagonals from corresponding vertices divide the polygons into triangles that are similar and similarly placed ..... Th. 79, p. 347
4. The ratio of corresponding segments equals the ratio of similitude ..... Th. 80, p. 349; Ex., p. 349; and §346, p. 350
5. The ratio of the areas equals the square of the ratio of similitude ..... Th. 82, p. 351; Th. 83, p. 352

RATIO AND PROPORTION

Two ratios are equal when:

1. A line is parallel to the base of a triangle. Th. 61, p. 260; §281, p. 261
2. Parallels cut two transversals ..... §282, p. 262
3. Polygons are similar ..... §291, p. 271; §297, p. 274
4. Two ratios are equal to a third ..... §296, p. 274

‡Optional.

Two products are equal when ratios obtained from them are equal.....§296, p. 274

Important cases of equal ratios occur when:

1. Two chords intersect within a circle.....Th. 66, p. 276
2. Two secants intersect without a circle.....Th. 67, p. 277
3. A secant and a tangent intersect without a circle...Th. 68, p. 279
4. A perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse.....Th. 69, p. 286

‡5. A line bisects an angle of a triangle.....Th. 63, p. 269

‡6. A line bisects an exterior angle of a triangle.....Th. 64, p. 270

Construction of proportional segments:

1. The division of a segment into parts proportional to any number of given segments..Prob. 14, p. 263; ‡Ex. 1, p. 264
2. The fourth term of a proportion when the other three are given in order.....Prob. 15, p. 265
3. The mean proportional to two given segments...Prob. 16, p. 288

AREAS OF POLYGONS

Measurement of polygons:

1. The area of a rectangle is hb.....Th. 71, p. 313; §326, p. 317
2. The area of a parallelogram is hb.....Th. 72, p. 316; §326, p. 317
3. The area of a triangle is $1/2\ hb$.....Th. 73, p. 317; §326, p. 317
4. The area of a trapezoid is $1/2\ h(b+b')$.....Th. 74, p. 318

‡5. For the area of irregular polygons see.....§329, p. 326

Polygons cover equal surfaces if:

1. They are made up of parts congruent in pairs.....Post. 46, p. 311, and §319, p. 312
2. They are parallelograms or triangles with equal bases and equal altitudes.....§330, p. 328
3. They are sums, differences, equal multiples, or equal parts of figures with equal surfaces.....§331, p. 328

Construction of polygons with equal surfaces:

‡1. To transform a polygon into a triangle.....Prob. 17, p. 330

‡2. To construct a square equal to the sum of two squares.....Prob. 18, p. 336

‡3. By algebraic analysis.....A Exercises, pp. 338–339

Pythagorean Theorem.....Th. 70, p. 289; ‡Th. 75, p. 334

‡Optional.

C Exercises—Oral Review

1. (*a*) What is a ratio? a proportion? (*b*) What is meant by the mean proportional between two segments or two numbers?

2. What is the mean proportional between 4 and 16? between a and b?

3. How would you find geometrically a segment equal to the $\sqrt{2}$ if a segment equal to 1 is given?

4. Write the proportion $\frac{27}{36}=\frac{15}{20}$ by mean alternation; by addition. What is the value of the ratios obtained in each case?

5. If $\frac{a}{b}=\frac{x}{y}$, what transformations would be used to obtain $\frac{a+b}{x+y}=\frac{b}{y}$.

6. Quote the theorems used to prove triangles similar.

7. For what purpose are similar triangles used in geometry? For what purpose are congruent triangles used?

8. Give an example of a practical application of similar figures.

9. If a, b, and c are three given segments, how would you construct a segment x if $x=\sqrt{ab}$? if $x=\frac{ac}{b}$?

10. What methods are used to prove ratios equal?

11. How do you prove two products equal?

12. If you are to use similar triangles to prove $\frac{XA}{XB}=\frac{YA}{YB}$ what triangles would you use?

13. Under what circumstances is the ratio of similitude of two similar figures equal to one?

14. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, what pairs of similar triangles are formed?

15. Give the properties of similar figures.

16. What two conditions are necessary in order to prove any two polygons similar?

17. Do similar polygons ever have the same area? When? Why?

18. If the areas of two similar triangles are 9 and 16 respectively and the base of the first is 5, what is the base of the second?

19. State briefly how the proof of the theorem concerning the area of a trapezoid is obtained.

20. Answer question 19 for (*a*) a triangle; (*b*) a parallelogram.

21. State a formula for the base of a rectangle if you have given A, the area, and h, the altitude.

22. Answer question 21 for a triangle instead of a rectangle.

23. State a formula for the altitude of a trapezoid if you have given A, the area, b and b', the bases, and h, the altitude.

24. How would you construct the product of two segments?

25. Construct two parallelograms with the same areas but different perimeters.

Using b and b' as bases and h and h' as altitudes, what is ratio of the areas of two parallelograms if

26. $b = b' = 12$, $h = 16$, and $h' = 20$?

27. $b = 15$, $b' = 20$, and $h = h' = 15$?

28. $b = 10$, $b' = 12$, $h = 8$, and $h' = 6$?

29. If the base and altitude of one rectangle are 18 and 24 respectively, what is the altitude of a rectangle that has the same area as the first if its base is 16?

30. What would be the altitude of the new rectangle in Ex. 29 if it is to be similar to the first?

31. What is the area of an isosceles right triangle if the hypotenuse is 6 ft.?

B Review Exercises

NOTE TO THE TEACHER. The following exercises are suitable for review work in proportion, similar figures, and areas. Work marked optional is in no way involved.

1. If two intersecting segments lie between two parallels, the triangles formed by the segments and the parallels are similar.

2. In Fig. 508, $XY \parallel AB$. If $AB = 18$, $AC = 30$, $CB = 20$, and $AX = 6$, find CY and XY.

3. If, in Fig. 508, $XY \parallel AB$ and $AB = 18$, $AC = 27$, $BC = 15$, and $XY = 12$, find CX and CY.

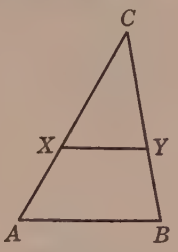


FIG. 508

4. If the diagonals of a quadrilateral divide each other into segments that have equal ratios, the figure has two parallel sides. Is this always true?

5. In Fig. 509, $ABCD$ is a square. Find pairs of congruent triangles and of similar triangles formed if $AE = BF = CG = DH$ and the points are joined as indicated. Read the ratios between corresponding sides of the similar triangles. If $AB = 36$ cm. and $AE = \frac{1}{3}AB$, find AX, XY, and YF.

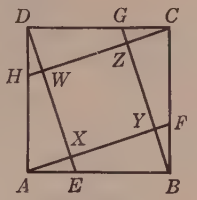


FIG. 509

6. In Ex. 5 find the area of $\triangle AEX$ and of $\triangle ABY$.

7. Prove by proportion that a line parallel to the bases of a trapezoid and bisecting one side bisects the other side also.

8. If, in Fig. 510, $AB \cdot AD = AC \cdot AE$ and $\angle B =$ a rt. $\angle$, prove $ED \perp AC$.

9. The legs of an isosceles triangle are 35 and the base is 56. Find the altitude and the area.

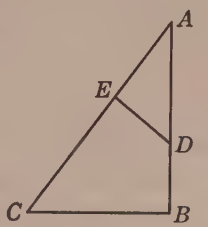


FIG. 510

10. The area of an isosceles triangle is 240. If the base is 20, find the altitude and the legs.

11. $ABCD$ is a parallelogram. E and F are the midpoints of AB and AD respectively. Prove that CF, CA, and CE divide the parallelogram into four parts that have equal areas.

12. Show how to divide a parallelogram into six equal parts by lines drawn from the same vertex.

13. Show how to divide a parallelogram into three equal parts by lines drawn from the same vertex.

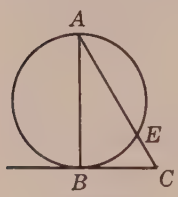


FIG. 511

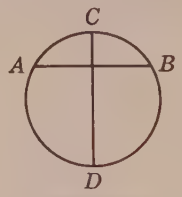


FIG. 512

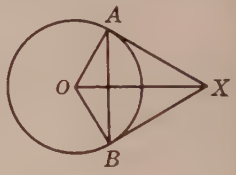


FIG. 513

14. Show that Th. 69, p. 286, may be made to apply in Fig. 511. AB is a diameter and BC a tangent at B. Point E is on the circle. Read the equal ratios obtained.

15. Show that Th. 69, p. 286, may be made to apply in Fig. 512 and read the ratios obtained. CD is a diameter perpendicular to the chord AB.

16. Show that Th. 69, p. 286, may be made to apply in Fig. 513 and read the equal ratios obtained. O is the center of the circle. AX and BX are tangents to the circle from X. A and B are the points of tangency.

17. If, in Fig. 514, AC is a mean proportional between AB and AO, prove $\triangle AOC \sim \triangle ABC$.

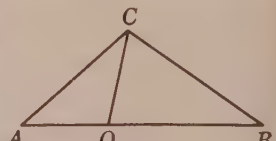


FIG. 514

18. $ABCD$ is any parallelogram. Draw the altitude DX. Construct on DX as base an isosceles triangle with the same area as the parallelogram.

19. The area of a square is one-half the area of the square on its diagonal.

20. If b and b' are the bases of two triangles and h and h' the corresponding altitudes, show that $\frac{h}{h'}=\frac{b}{b'}$ if the triangles are similar and $\frac{h}{h'}=\frac{b'}{b}$ if they have equal areas.

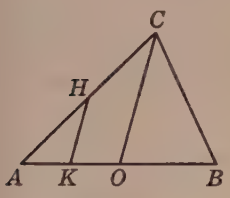


FIG. 515

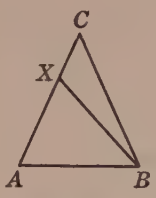


FIG. 516

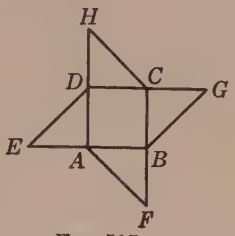


FIG. 517

21. In Fig. 515, O is the mid-point of AB, $AK=\frac{1}{4}AB$, and $HK \parallel CO$. What is the ratio of CH to CA? Why?

22. In Fig. 516, $\triangle ABC$ is isosceles. If AB is a mean proportional between AC and AX, prove $\triangle ABX$ isosceles.

23. Find the area of a square if the diagonal is 15 in.

24. In Fig. 517, $ABCD$ is a square. Isosceles right triangles are constructed on the sides as shown. Find the total area if AB is 6 in.

25. If, in Fig. 517, $AF=6$ in. find the total area and the perimeter of the figure. $ABCD$ is a square with isosceles right triangles constructed on the sides.

26. The product of two legs of a right triangle is equal to the product of the hypotenuse and the altitude on the hypotenuse.

27. $ABCD$ is a trapezoid with the altitude 15 in. and the bases 24 in. and 36 in. respectively. What will be the altitude of a triangle with the same area whose base is equal to the altitude of the trapezoid?

A Review Exercises

1. The altitude on the hypotenuse of a right triangle divides it into two triangles whose areas have the same ratio as the squares on the legs of the right triangle.

2. If the segments which join the vertices of a polygon with a given point are divided in the same ratio from the given point and the points of division joined in the same order as the vertices of the polygon, the polygon so formed and the given polygon are **radially** placed. The point may be without or within the polygon. In each case $\frac{OA}{OA'} = \frac{OB}{OB'} = \text{etc.}$ The radial point is called the **center of similitude** of the polygons (Fig. 518). Draw the figure when point O is within the polygon.

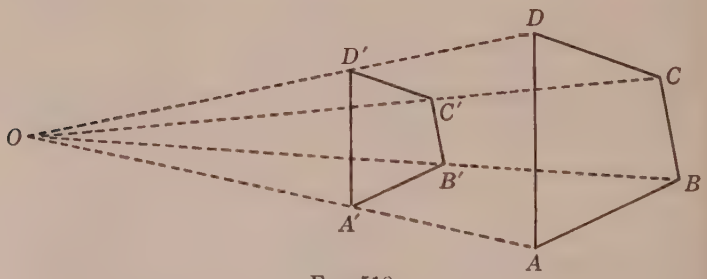


FIG. 518

3. Two polygons that are radially placed are similar. Give the proof for the two cases.

4. Show how a polygon may be constructed similar to a given polygon $ABCD$ with a given segment $A'B'$ corresponding to side AB and with any arbitrary point O as a center of similitude.

NOTE. The center of similitude of two similar figures may be of use in enlarging or reducing drawings.

5. Using any three given segments a, b, c, construct a rectangle whose sides are $(a+b)$ and c. Show how this figure illustrates the algebraic identity $c(a+b) = ac+bc$.

Suggestion. $c(a+b)=ac+bc$ may be translated into geometry: The rectangle whose sides are c and $(a+b)$ may be divided into two rectangles whose sides are a and c, and b and c, respectively.

6. Illustrate geometrically the identity $(a-b)^2=a^2+b^2-2ab$.

Suggestion. Translate $(a-b)^2=a^2+b^2-2ab$ into geometry. Construct a figure formed by adding a square whose side is a to a square whose side is b. Show how two rectangles may be cut from this so as to leave the desired result. What will be the sides of these rectangles?

7. Translate into geometry and illustrate by a figure:

a. $c(a-b)=ca-cb$.
b. $(a+b)^2=a^2+b^2+2ab$.
c. $(a+b)(c+d)=ac+bc+ad+bd$.
d. $(a^2-b^2)=(a-b)(a+b)$.

8. In Fig. 519, $ES=93{,}000{,}000$ miles, $BE=4000$ miles, and AS is 109 times as much. Find EO. AO and DO are tangent to both circles.

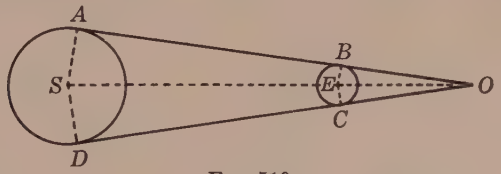


FIG. 519

NOTE. Fig. 519 is used in astronomy. S represents the sun, E the earth, and $EBOC$ the shadow of the earth. The numbers given are approximately correct.

‡**9.** Prove the converse of Th. 63 and of Th. 64.

10. In Fig. 520, $ABCD$ is a square. D is the center and DB is the radius for the arc BE; A is the center and AE is the radius for the arc EF; D is the center and DF is the radius for the arc FG, etc. Find the length of DE, AF, DG, AH, etc., if $AB=1$.

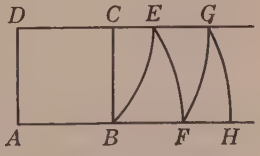


FIG. 520

‡ Depends upon optional work.

11. If, in Fig. 521, $AB = AC = 1$, show that $BC = \sqrt{2}$. If $AD = BC$, what is the length of BD? If $AE = BD$, what is the length of BE? Show how the figure may be continued so as to construct segments equal to $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc.

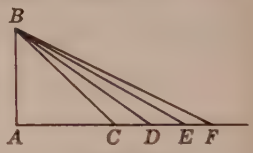


FIG. 521

12. If similar polygons are constructed on the sides of a right triangle as corresponding sides, the area of the polygon constructed on the hypotenuse is equal to the sum of the areas of the polygons on the other two sides (see Fig. 522).

Suggestion: To prove that $C = A + B$, find $\frac{A}{C}$ and $\frac{B}{C}$ in terms of the sides of the triangle, and add.

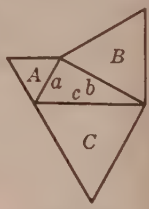


FIG. 522

13. *If two triangles have their corresponding sides parallel each to each, the triangles are similar* (Fig. 523).

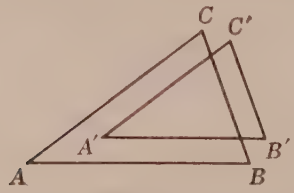


FIG. 523

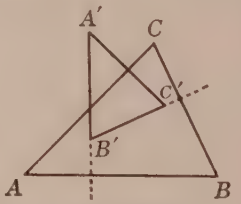


FIG. 524

Give an indirect proof. We have the following possibilities (see Th. 16, p. 99).

a. $A + A' = 2$ rt. $\measuredangle$, $B + B' = 2$ rt. $\measuredangle$, $C + C' = 2$ rt. $\measuredangle$.

b. $A + A' = 2$ rt. $\measuredangle$, $B + B' = 2$ rt. $\measuredangle$, $C = C'$.

c. $A + A' = 2$ rt. $\measuredangle$, $B = B'$, $C = C'$.

d. $A = A'$, $B = B'$, $C = C'$.

14. *If two triangles have their corresponding sides perpendicular each to each, the triangles are similar* (Fig. 524).

The proof is similar to that for Ex. 13 (see Th. 19, p. 110).

15. In Fig. 525, ABC is any triangle. $ACED$ and $BFEC$ have the common side CE and have AC and BC as bases. Prove that if DF is joined, $ABFD$ is a parallelogram whose area is equal to the sum of the areas of $ACED$ and $BFEC$.

Suggestion. Extend EC to meet AB.

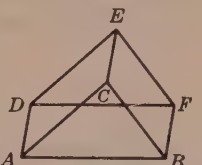


FIG. 525

16. In Fig. 526, ABC is any triangle. $ACFG$ and $BCED$ are any parallelograms on the sides AC and BC respectively. GF and DE are extended to meet at M. On AB a parallelogram is constructed with one side AH equal and parallel to MC. Prove $\square ABKH$ has the same area as $\square ACFG + \square BCED$.

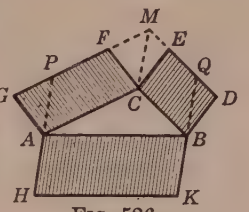


FIG. 526

NOTE. Ex. 16 is known as Pappus' Theorem (about 300 A.D.).

17. Deduce the Pythagorean Theorem from Ex. 16.

18. Inscribe a square in a given triangle.

Suggestion. The construction is suggested in Fig. 527. To prove $HG = HE$, prove $\frac{DC}{HG} = \frac{DW}{HE}$.

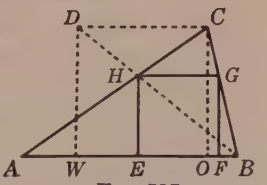


FIG. 527

19. Solve Ex. 18 by the construction shown in Fig. 528.

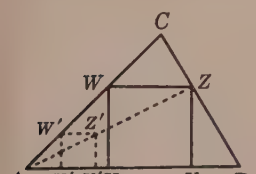


FIG. 528

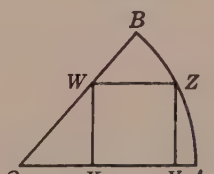


FIG. 529

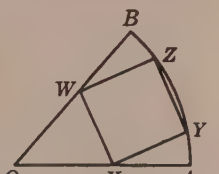


FIG. 530

20. Show how to inscribe a square in a sector (*a*) as shown in Fig. 529, (*b*) as shown in Fig. 530.

UNIT IX

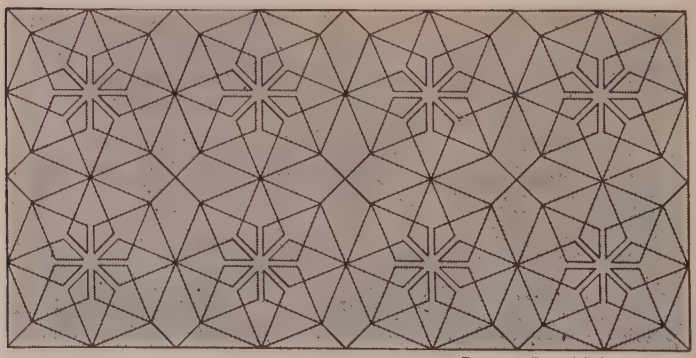
REGULAR POLYGONS

350. Regular polygons in the world about you. Regular polygons abound. They are found in nature, art, and industry, in trademarks and other forms of advertising. Towers, spires, bay and dormer windows, hoppers and nuts may be hexagonal or octagonal in form. Many snow crystals are hexagonal in structure (Fig. 162, p. 114). The cells of the wasp's nest and of the comb in the beehive are hexagonal (Fig. 534, p. 371). The starfish is usually pentagonal in structure (Fig. 536, p. 373). One common linoleum pattern consists of regular octagons and squares (Fig. 546, p. 383). Show that the design below is based on regular octagons.

351. A **regular polygon** has been defined as a polygon with all of its sides and all of its angles equal.

Ex. 1. *A regular quadrilateral is a square.*

Ex. 2. *An equilateral triangle is a regular polygon.*



Courtesy Pascal Sylvester, Inc.

FIG. 531. **Orna-metal brass strips for art terrazzo floors**

THEOREM 84*

352. A circle can be circumscribed about any regular polygon.

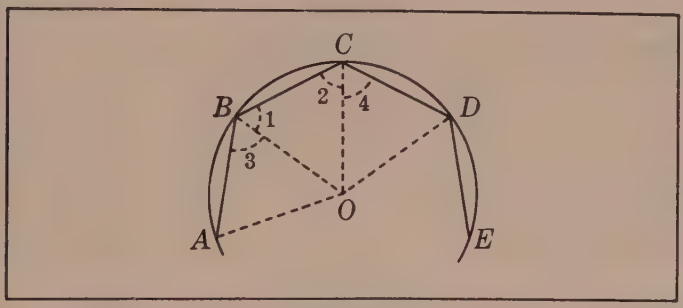


FIG. 532

Hypothesis: $ABCDE$ etc. is any regular polygon.

Conclusion: A circle can be circumscribed about polygon $ABCDE$ etc.

Analysis and construction:

1. To prove that a circle can be circumscribed about polygon $ABCDE$ etc., prove that a circle can be drawn that will pass through the vertices A, B, C, D, E, etc.
2. $\therefore$ pass $\odot O$ through points A, B, and C and prove that it will pass through D, E, etc.
3. To prove that $\odot O$ will pass through D, draw OD, OA, OB, and OC and prove $OD = OA$.
4. To prove $OD = OA$, prove $\triangle AOB \cong \triangle DOC$.
5. $\therefore$ prove $OB = OC$, $\angle 3 = \angle 4$, and $AB = CD$.

353. The radius of the circumscribed circle of a regular polygon is called the **radius** of the polygon.

354. COR. **A radius of a regular polygon bisects the angle through which it passes.**

THEOREM 85

355. A circle can be inscribed in any regular polygon.

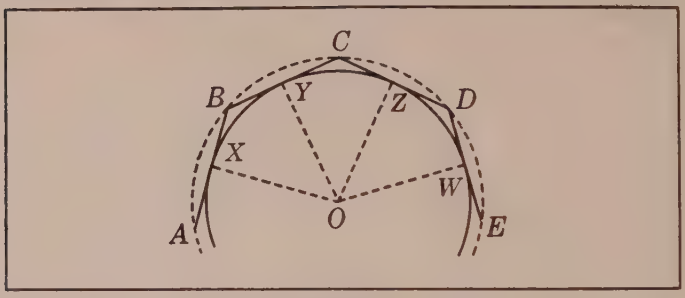


FIG. 533

Hypothesis: $ABCDE$ etc. is any regular polygon.

Conclusion: A circle can be inscribed in polygon $ABCDE$ etc.

Analysis and construction:

1. To prove that a circle can be inscribed in polygon $ABCDE$ etc., prove that a circle can be drawn tangent to the sides of $ABCDE$ etc.
2. $\therefore$ prove that a point can be found which is equally distant from AB, BC, CD, etc.
3. $\therefore$ circumscribe $\odot O$ about polygon $ABCDE$ etc., draw OX, OY, OZ, etc., from O perpendicular to AB, BC, CD, etc., respectively, and prove OX, OY, OZ, etc., radii.
4. $\therefore$ prove $OX = OY = OZ =$ etc. (see Th. 44, p. 199.)

356. The center of the circumscribed and of the inscribed circle of a regular polygon is called the **center** of the polygon.

357. The radius of the inscribed circle of a regular polygon is called the **apothem** of the polygon.

358. Cor. I. **The apothem of a regular polygon is perpendicular to the sides.**

359. By the **central angle** of a regular polygon is meant the angle between two consecutive radii.

360. Cor. II. **The central angle of a regular polygon of n sides is $^1/_n$ of 360°.**

361. Review Th. 20, p. 112. What is the formula for each angle of a regular polygon of n sides?

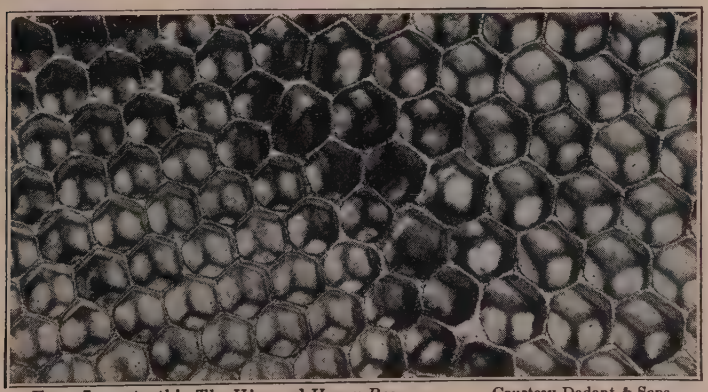
Exercise. Make a table showing the number of degrees in the angle at the center and in one vertex angle of regular polygons of 3, 4, 5, 6, 8, 10, 12, 15, 16, 24, and 32 sides.

B Exercises

1. The central angle of a regular polygon is the supplement of an angle of the polygon.

2. *The radius of a regular polygon bisects the angle between two consecutive apothems.*

3. The apothem of a regular polygon bisects the angle between two consecutive radii and the minor arc cut off by the side to which it is drawn.



From Langstroth's *The Hive and Honey Bee* Courtesy Dadant & Sons

Fig. 534. **Brood-combs of the hive bee with cells containing worker grubs.** Note the hexagonal cells.

THEOREM 86

362. If a circle is divided into n equal arcs,

A. The chords joining the points of division form an inscribed regular polygon.

B. The tangents drawn to the points of division form a circumscribed regular polygon.

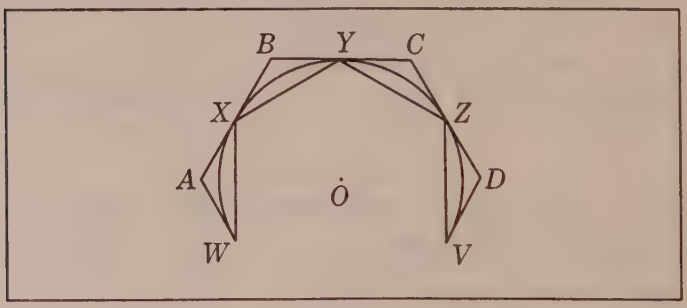


FIG. 535

A. Hypothesis: Circle O is divided into the equal arcs WX, XY, YZ, etc. Chords WX, XY, YZ, etc., join the points of division.

Conclusion: $WXYZ$ etc. is a regular polygon.

Analysis:

1. To prove $WXYZ$ etc. a regular polygon, prove
   a. $WX = XY = YZ =$ etc. (see Th. 41B, p. 195).
   b. $\angle WXY = \angle XYZ = \angle YZV =$ etc.
2. To prove $\angle WXY = \angle XYZ = \angle YZV =$ etc., show that each angle is measured by $\frac{1}{2}(n-2)$ equal arcs.

Exercise. (*Optional.*) Show that each of the n equal arcs is equal to $\frac{360°}{n}$. From this find the value of the $(n-2)$ equal arcs and of each angle of the polygon.

B. Hypothesis: Circle O is divided into the equal arcs WX, XY, YZ, etc. Tangents AB, BC, CD, etc., are drawn to the points of division X, Y, Z, etc.

Conclusion: $ABCD$ etc. is a regular polygon.

Analysis and construction:

1. To prove $ABCD$ etc. a regular polygon, prove
   a. $AB = BC = CD =$ etc.
   b. $\angle A = \angle B = \angle C =$ etc.
2. $\therefore$ join WX, XY, YZ, ZV, etc., and prove $\triangle WXA \cong \triangle XYB \cong \triangle YZC \cong$ etc.

Suggestion. Notice that

$AX = BY = CZ =$ etc., by congruent triangles.

$BX = CY = ZD =$ etc., by congruent triangles.

$\therefore AB = BC = CD =$ etc., by addition.

Courtesy W. K. Fisher

FIG. 536. **The rays of the commonest form of the starfish resemble lines drawn from the center to the vertices of a pentagon.**

Problem 20

363. To inscribe a square in a circle.

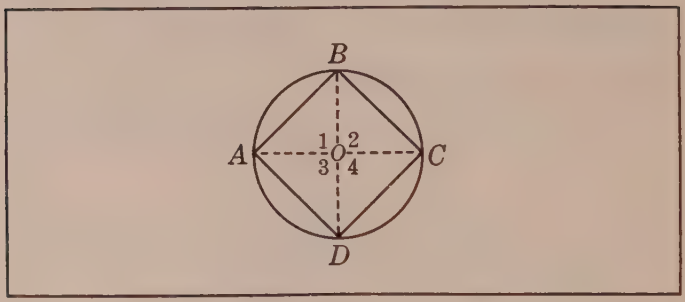


Fig. 537

Given circle O.

To inscribe a square in circle O.

Analysis:

1. To inscribe a square in $\odot O$, inscribe a regular 4-side in $\odot O$ (use Th. 86A, p. 372).
2. $\therefore$ divide the circle into four equal arcs.
3. $\therefore$ construct four equal central angles at point O.

Directions: Draw AC and BD two perpendicular diameters and draw AB, BC, CD, and DA. Give proof.

364. Cor. I. **By drawing two perpendicular diameters and drawing tangents to the circle at the ends of the diameters, a square may be circumscribed about a circle.**

365. Cor. II. **By bisecting the minor arcs cut off by the sides of the inscribed square, the circle is divided into eight equal arcs. The chords joining the eight points of division form an inscribed regular octagon, and tangents drawn to the eight points of division form a circumscribed regular octagon.**

Problem 21

366. To inscribe a regular hexagon in a circle.

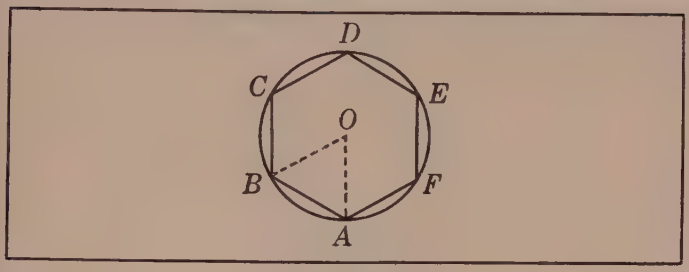


Fig. 538

Given circle O.

To inscribe a regular hexagon in circle O.

Analysis:

1. To inscribe a regular hexagon in a circle, divide the circle into six equal arcs.
2. $\therefore$ construct $\overset{\frown}{AB} = 60°$.
3. $\therefore$ construct $\angle AOB = 60°$.

Directions:

1. Construct $\triangle AOB$ equilateral.
2. Construct six arcs, each equal to $\overset{\frown}{AB}$, and join the points of division. Give proof.

367. Cor. I. **By joining alternate vertices of an inscribed regular hexagon, starting with any vertex, an inscribed equilateral triangle may be formed.**

368. Cor. II. **By bisecting the minor arcs cut off by the sides of the inscribed regular hexagon, the circle is divided into 12 equal arcs. The chords joining the points of division form an inscribed regular 12-side.**

369. Cor. III. **Circumscribed regular polygons of 3, 6, 12, etc., sides can be constructed.**

THEOREM 87*

370. The area of a regular polygon is one-half the product of the perimeter and the apothem.

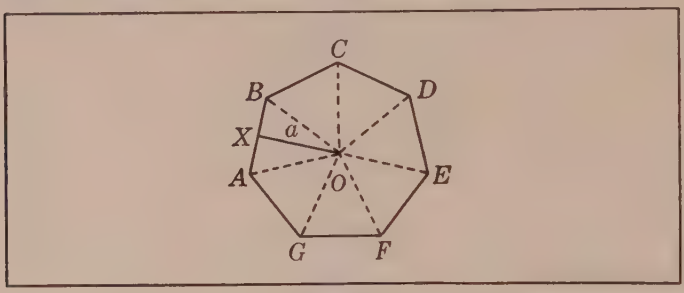


FIG. 539

Hypothesis: $ABCDE$ etc. is a regular polygon. Let p represent its perimeter and a its apothem.

Conclusion: Area $ABCDE$ etc. $= \frac{1}{2}ap$.

Analysis:

1. To prove area $ABCDE$ etc. $= \frac{1}{2}ap$, divide the polygon into triangles and add their areas.
2. $\therefore$ draw the radii and add the areas of $\triangle AOB$, BOC, COD, etc.

Outline of proof. Give the reasons.

1. Use the sides AB, BC, CD, etc., as bases of the triangles.
2. The altitude of each triangle is equal to the apothem of the polygon.
3. Area $AOB = \frac{1}{2}\, a \cdot AB$.
4. Area $BOC = \frac{1}{2}\, a \cdot BC$.
5. Area $COD = \frac{1}{2}\, a \cdot CD$.

   . . . . . etc.
6. $\therefore$ area$(AOB + BOC + COD + \text{etc.}) =$
   $\frac{1}{2}a(AB + BC + CD + \text{etc.})$.
7. $\therefore$ area $ABCDE$ etc. $= \frac{1}{2}ap$.

C Exercises

1. Show how to construct inscribed and circumscribed regular polygons of 16 sides; of 24 sides.

2. Show how to divide a regular hexagon into two congruent isosceles trapezoids; three congruent rhombuses; or six congruent equilateral triangles.

3. The apothem of a square is one-half its side.

4. The radius of a circle is 6 in. Find one side, the apothem, and the area of the inscribed square.

5. The radius of a circle is 6 in. Find the side, the apothem, and the area of the inscribed regular hexagon.

B Exercises

1. The radius of a circle is 6 in. Find one side, the apothem, and the area of the circumscribed regular hexagon.

2. ABC is an equilateral triangle inscribed in a circle. Prove the diameter from C is the perpendicular bisector of AB.

3. Any side of an inscribed equilateral triangle bisects the radius drawn to the mid-point of the minor arc cut off by that side.

4. The apothem of an equilateral triangle is one-half its radius.

5. The radius of a circle is 6 in. Find one side, the apothem, and the area of the inscribed equilateral triangle.

6. The radius of the inscribed equilateral triangle is the apothem of the circumscribed equilateral triangle.

7. The radius of the inscribed equilateral triangle is one-half the radius of the circumscribed equilateral triangle.

8. The radius of a circle is 6 in. Find one side, the apothem, and the area of the circumscribed equilateral triangle.

THEOREM 88

371. Two regular polygons of the same number of sides are similar.

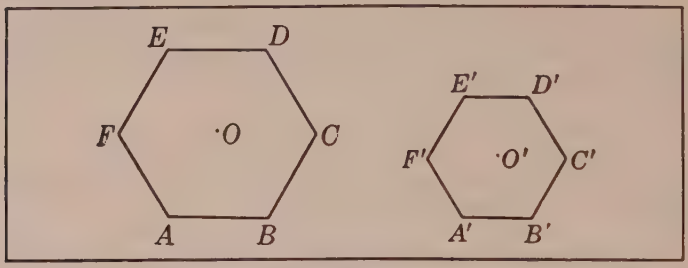


FIG. 540

Hypothesis: The two regular polygons O and O' have the same number of sides.

Conclusion: Polygon $O \sim$ polygon O'.

Analysis:

1. To prove polygon $O \sim$ polygon O', prove
   a. $\angle A = \angle A'$, $\angle B = \angle B'$, etc. (§132, p. 112).
   b. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.}$
2. To prove $\frac{AB}{A'B'} = \frac{BC}{B'C'}$, prove $\frac{AB}{BC} = \frac{A'B'}{B'C'} = 1$.

Ex. 1. The perimeters of two regular polygons of the same number of sides have the same ratio as two corresponding sides.

Ex. 2. The areas of two regular polygons of the same number of sides have the same ratio as the squares of two corresponding sides.

Ex. 3. Show how to divide two regular polygons of the same number of sides into a series of similar isosceles triangles. Can this be done in more than one way? Try regular polygons of different kinds, two regular 3-sides, 5-sides, 6-sides, 10-sides, etc. Can the triangles obtained in each polygon be proved congruent? Why?

THEOREM 89

372. If two regular polygons have the same number of sides, the ratio of the radii or of the apothems is equal to the ratio of any two corresponding sides.

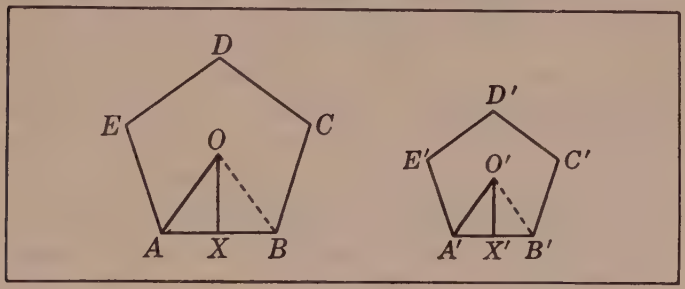


FIG. 541

Hypothesis: O and O' are two regular polygons of the same number of sides. OA, $O'A'$ are radii; OX, $O'X'$ apothems. AB and $A'B'$ are corresponding sides.

Conclusion: $\frac{OA}{O'A'} = \frac{AB}{A'B'} = \frac{OX}{O'X'}$.

Analysis:

1. To prove $\frac{OA}{O'A'} = \frac{AB}{A'B'}$, $\begin{cases} \text{draw } OB \text{ and } O'B', \\ \text{prove } \triangle AOB \sim \triangle A'O'B'. \end{cases}$
2. " " $\frac{OX}{O'X'} = \frac{AB}{A'B'}$, use Th. 80, p. 349.

373. COR. I. **In two regular polygons of the same number of sides, the ratio of the perimeters equals the ratio of the radii or of the apothems.**

374. COR. II. **The ratio of the perimeter of a regular polygon to the radius of the inscribed or of the circumscribed circle is the same for all regular polygons of the same number of sides.**

375. COR. III. **In two regular polygons of the same number of sides, the ratio of the areas is equal to the ratio of the squares of the radii or of the apothems.**

C Exercises

1. *An equilateral polygon inscribed in a circle is regular.*

2. The sides of two regular polygons of the same number of sides are 12 and 16 respectively. Find the ratio of the radii, the apothems, and the areas.

3. The areas of two regular polygons of the same number of sides are 144 and 121 respectively. Find the ratio of their radii.

4. If one side of a regular polygon is extended, the exterior angle formed is equal to the central angle.

5. From Ex. 4 find the sum of all the exterior angles of a regular polygon of n sides.

6. Construct a square on a given segment as diagonal.

7. On a segment as one side, construct a regular hexagon without constructing the circumscribed circle.

8. The radius of an inscribed regular polygon is the apothem of the regular polygon of the same number of sides circumscribed about the same circle.

9. The apothem of a regular polygon bisects the side to which it is drawn.

10. Find the area of a regular hexagon if one side is 10.

B Exercises

1. Find the ratio of one side of the inscribed square to one side of the square circumscribed about the same circle.

2. Find the ratio of the areas of the two squares mentioned in Ex. 1.

3. Find the ratio of one side of an inscribed equilateral triangle to one side of the equilateral triangle circumscribed about the same circle.

4. Find the ratio of the areas of the two triangles mentioned in Ex. 3.

5. Answer Ex. 3 for the inscribed and circumscribed regular hexagon.

6. Answer Ex. 4 for the inscribed and circumscribed regular hexagon.

7. The area of the inscribed equilateral triangle is half the area of the regular hexagon inscribed in the same circle.

8. The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

9. The diagonals of a regular pentagon are equal (Fig. 542).

10. Fig. 542 shows an inscribed regular pentagon with its diagonals. Prove

a. $\triangle AXB$, $\triangle AXW$, and $\triangle AYB$ are isosceles.

b. Polygon $VWXYZ$ is regular.

c. $\triangle ABC \sim \triangle AXB$. Write the ratios of corresponding sides.

d. $\triangle AXW \sim \triangle ABW$. Write the ratios of corresponding sides.

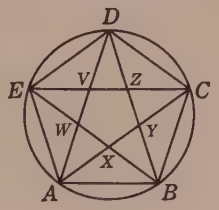


FIG. 542

11. Fig. 543 shows an inscribed regular hexagon with certain diagonals drawn. Prove AD perpendicular to EC.

12. In Fig. 543 prove that the diagonals EC and FB are parallel.

13. In Fig. 543 prove that $UVWXYZ$ is a regular hexagon.

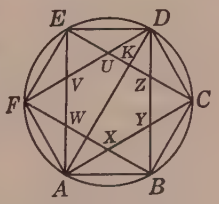


FIG. 543

14. If squares are described outwardly on the sides of a regular hexagon, the outer vertices of the squares are the vertices of a regular duodecagon.

15. A principal diagonal of a regular hexagon passes through the center of the circumscribed circle and is parallel to a pair of opposite sides.

16. If a regular polygon has an even number of sides, the diameter of the circumscribed circle drawn from any vertex passes through the opposite vertex.

17. If a regular polygon has an even number of sides, the opposite sides are parallel.

18. If any regular polygon has an odd number of sides, the diameter of the circumscribed circle drawn from any vertex is a perpendicular bisector of the opposite side.

A Exercises

If the radius of the circle is r, prove that:

1. For the inscribed square, one side is $r\sqrt{2}$, the apothem is $\frac{1}{2}r\sqrt{2}$, and the area is $2r^2$.

2. For the inscribed regular hexagon, one side is r, the apothem is $\frac{1}{2}r\sqrt{3}$, and the area is $\frac{3}{2}r^2\sqrt{3}$.

3. For the inscribed equilateral triangle, one side is $r\sqrt{3}$, the apothem is $\frac{1}{2}r$, and the area is $\frac{3}{4}r^2\sqrt{3}$.

4. On a given segment as a side of a regular octagon, construct the circumscribed circle.

Suggestion. Construct an isosceles triangle with the given segment as base and the vertex angle 45°.

5. Show that a regular octagon may be inscribed in a square by the following method (Fig. 544): With O as center and OH, the half median, as radius cut the diagonals at W, X, Y, and Z. At W, X, Y, and Z construct perpendiculars to the diagonals.

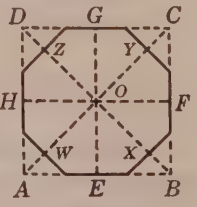


FIG. 544

6. Show that a regular octagon may be inscribed in a square by the following method (Fig. 545): With the vertices of the square as centers and the semi-diagonal as radius cut the sides of the square at X, Y, Z, etc., and join the points as indicated.

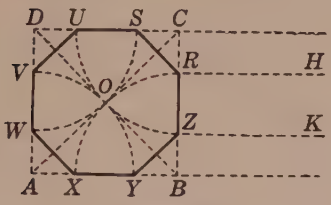


FIG. 545

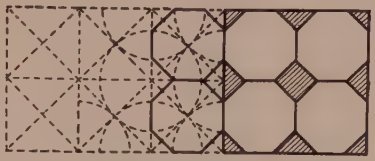


FIG. 546

NOTE. Exs. 5 and 6 are extremely useful. They can be used as shown in Fig. 546 to construct a tiled floor pattern composed of regular octagons and squares, or as in Fig. 545 to cut a square timber down into an octagonal one. The cuttings at the sawmill would be along the lines YZ and ZK, SR and RH, and so on around the timber. Ex. 6 gives the common method for cutting out the pattern for an octagonal table top.

7. Given the secondary diagonal AB of a regular octagon (Fig. 547), to construct the octagon.

Suggestion. Since the number of degrees in $\angle 1$ is known, $\triangle AEB$ may be constructed. The problem is then to construct $DC \parallel AB$ so that $DC = AD$.

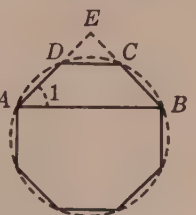


FIG. 547

NOTE. Ex. 7 is a problem which might occur in building octagonal bay windows. $ABCD$ would be the outline of the window and AB the line on the house.

8. $ABCDEF$ is a regular hexagon. X, Y, and Z are the mid-points of AF, BC, and DE respectively. Prove that XYZ is an equilateral triangle and find its area if AB is equal to 20 in.

PROBLEM 22 (*Optional*)

376. To divide a given segment so that the larger portion is a mean proportional between the whole segment and the smaller portion.

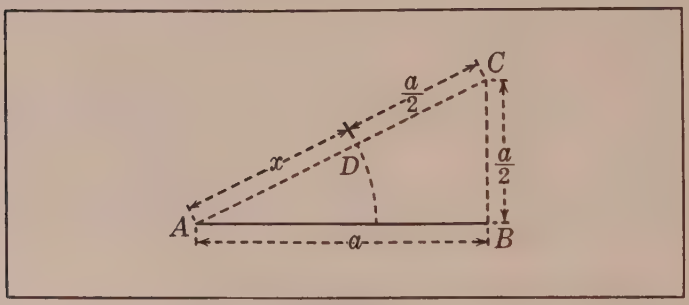


FIG. 548

Given segment $AB = a$.

To find a segment x so that $\frac{a}{x} = \frac{x}{a-x}$.

Analysis and construction:

1. To find x so that $\frac{a}{x} = \frac{x}{a-x}$, solve for x.

$$x^2 = a^2 - ax.$$
$$x^2 + ax = a^2.$$
$$x^2 + ax + (\tfrac{1}{2}a)^2 = a^2 + (\tfrac{1}{2}a)^2.$$
$$(x + \tfrac{1}{2}a)^2 = a^2 + (\tfrac{1}{2}a)^2.$$

The last equation suggests the Pythagorean Theorem.

2. $\therefore$ construct a right triangle with a and $\frac{1}{2}a$ for perpendicular sides. The hypotenuse will be $x + \frac{1}{2}a$.

3. $\therefore$ x will be the difference between the hypotenuse and $\frac{1}{2}a$.

Outline of proof. Give the reasons.

1. $(x+\frac{1}{2}a)^2=a^2+(\frac{1}{2}a)^2$.
2. $x^2+ax+(\frac{1}{2}a)^2=a^2+(\frac{1}{2}a)^2$.
3. $x^2+ax=a^2$.
4. $x^2=a^2-ax$.
5. $x^2=a(a-x)$.
6. $\frac{a}{x}=\frac{x}{a-x}$.

377. If a segment is so divided that the larger portion is a mean proportional between the whole segment and the smaller portion, the segment is said to be divided into **extreme and mean ratio.** Problem 22 may be stated: To divide a segment into extreme and mean ratio.

NOTE. The division of a segment into extreme and mean ratio is often called the **golden section.** It was known to the Pythagoreans. It is an interesting fact, often used in the theory of design, that that division of a rectangle which is the most pleasing to the majority of people is the one that most closely approximates the golden section. It might be used in designs for doors or windows (Fig. 549).

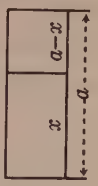


FIG. 549

A Exercises

1. If a segment 3 in. long is divided into mean and extreme ratio, find the length of each part.

Suggestion. Solve the equation $\frac{3}{x}=\frac{x}{3-x}$. The resulting equation $x^2+3x-9=0$ cannot be factored. It is necessary, therefore, to solve it by some other method. See §422, p. 428. The result contains the $\sqrt{5}$. The golden section always involves $\sqrt{5}$, and is used in the construction of the pentagon and decagon. For the construction of these figures see pp. 386–388.

2. Show that in $\frac{a}{x}=\frac{x}{a-x}$, the ratio $\frac{a}{x}$ is approximately 13 : 8.

PROBLEM 23 (*Optional*)

378. To inscribe a regular decagon in a circle.

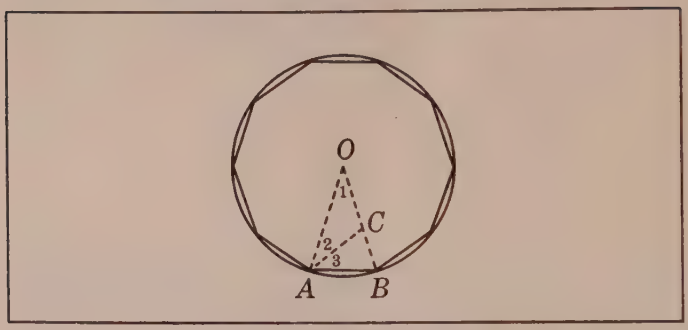


FIG. 550

Given circle O.

To construct an inscribed regular decagon.

Analysis:

1. *a.* To construct AB, a side of the decagon, construct $\angle O = \frac{1}{10}$ of 4 rt. $\measuredangle$ or $\frac{1}{5}$ of 2 rt. $\measuredangle$.
   b. $\therefore$ construct $\triangle ABO$ isosceles with $\angle A = 2\angle O$.
2. *a.* Since $\angle A = 2\angle O$, try bisecting $\angle A$. Then $\triangle ACO$ would be isosceles and $\triangle ABC \sim \triangle ABO$.
   b. Then $\frac{OB}{AB}$ would equal $\frac{AB}{BC}$.
   c. Then $\frac{OB}{OC}$ would equal $\frac{OC}{CB}$.

Construction:

1. Divide the radius of the circle, OB, into extreme and mean ratio at C.
2. Use the larger segment, OC, as a side of the decagon.
3. $\therefore$ make $AB = OC$. Draw OA and prove $\angle O = \frac{1}{5}$ of a straight angle.

Outline of proof. Give the reasons.

1. $\frac{OB}{OC}=\frac{OC}{CB}$.
2. $\frac{OB}{AB}=\frac{AB}{BC}$.
3. $\triangle ABC \sim \triangle ABO$.
4. $\triangle ABO$ is isosceles.
5. $\therefore \triangle ABC$ is isosceles.
6. $AC=AB=OC$.
7. $\therefore \triangle AOC$ is isosceles.
8. $\therefore \angle 2=\angle O$.
9. $\angle 3=\angle O$.
10. $\therefore \angle A=2\angle O$.
11. $\angle A+\angle B+\angle O=5\angle O$.
12. $\therefore \angle O=\frac{1}{5}\ 180°=\frac{1}{10}\ 360°$.

379. COR. I. (*Optional.*) **A regular pentagon may be inscribed in a circle** (Fig. 551).

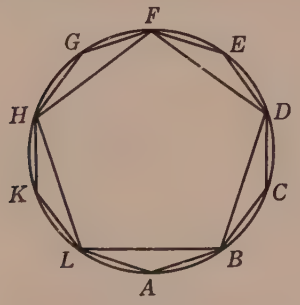


FIG. 551

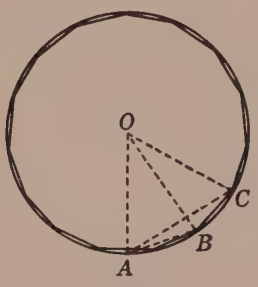


FIG. 552

380. COR. II. (*Optional.*) **A regular pentadecagon may be inscribed in a circle.**

Suggestion. The central angle must be 24°. Be sure to note that $24°=60°-36°$ (Fig. 552).

NOTE. **Construction of other regular polygons.** From the time of Euclid until 1796 it was supposed that the regular polygons mentioned in this unit were the only ones that could be constructed with ruler and compasses. This includes polygons with $3 \cdot 2n$, $4 \cdot 2n$, $5 \cdot 2n$, and $15 \cdot 2n$ sides. The smaller polygons not included in this set are those of 7, 9, 11, 13, 14, 17, 18, 19 sides. In 1796, however, Karl Friedrich Gauss, then a young man of nineteen in one of the German universities, proved that regular polygons with a prime number of sides could be inscribed in a circle by means of ruler and compasses if and only if the prime number was of the form 2^n+1. That no polygons of 7, 9, 11, 13, 19, etc., sides can be constructed in the given manner follows from Gauss's proof. A polygon of 7 sides can be constructed by the use of a parabola and a circle; one of 9 sides by the use of a hyperbola and a circle.

While regular polygons of 7, 9, 11, 13, etc., sides cannot be constructed exactly with ruler and compasses, methods for constructing them approximately are frequently given in courses in mechanical drawing. The accuracy of these methods can be tested by trigonometry. Such polygons are sometimes used in advertising and in ornament.

381. Star polygons. (*Optional.*) Reference has been made to star polygons on page 113 and on page 252 in Exs. 15 and 16 and the note following. There are one or more regular star polygons related to each of the convex regular polygons. They have been studied wherever geometry has been studied. Fig. 553 shows one formed by dividing the circle into sixteen equal parts and joining each point to the seventh from it. Let the pupil make drawings of other regular star polygons.

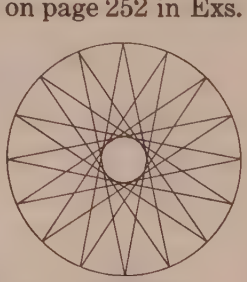


FIG. 553

The following questions refer to Fig. 553:

Ex. 1. How many sets of equal angles are there in the figure?

Ex. 2. Find the number of degrees in one angle of each set.

Ex. 3. How many sets of equal segments are there in the figure?

Ex. 4. How many regular polygons of 16 sides can be formed by joining corresponding intersections? Give proof for one.

Ex. 5. How many regular octagons can be formed by joining corresponding intersections? Give proof for one.

Ex. 6. How many squares are there in the figure?

A Exercises

1. What inscribed and circumscribed regular polygons can be obtained from the inscribed regular decagon? How? From the inscribed regular pentadecagon? How?

2. How can you obtain regular star polygons from the inscribed regular 20-side?

3. Show that the radius of a circle may be divided into extreme and mean ratio and therefore that a side of the inscribed regular decagon may be obtained as follows: Draw AB and CD, two diameters perpendicular to each other. Let E be the mid-point of OA. With E as center and EC as radius cut OB at F. OF is a side of the decagon (Fig. 554).

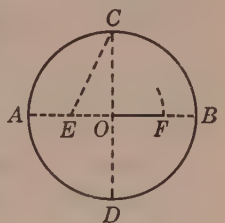


FIG. 554

4. Fig. 555 shows an inscribed regular pentagon with its diagonals. Prove that

a. AC is divided into extreme and mean ratio at Y.

Suggestion: Prove $\frac{AC}{CB}=\frac{CB}{CY}$.

b. AY is divided into mean and extreme ratio at X.

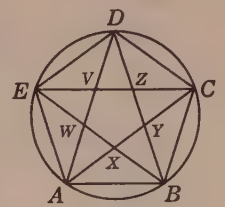


FIG. 555

5. From the foregoing exercises show how to construct the diagonal of a regular pentagon, given one side, and therefore how to construct the regular pentagon, given one side.

6. Show that if AD is taken as the radius of a circle, AB will be a side of the inscribed regular decagon (Fig. 555).

7. The segments joining the mid-points of the sides of a regular pentagon in order form a regular pentagon.

Survey Test

1. Define a regular polygon. Name as many regular polygons as you can.

2. Name the properties of regular polygons.

3. What is the radius of a regular polygon? What fact or facts do you know about it?

4. What is the center of a regular polygon? What is its apothem? What fact or facts do you know about the apothem?

5. (*a*) Define the central angle of a regular polygon. (*b*) What is its measure? (*c*) State the formula for the measure of an angle of a regular polygon.

6. Quote three reasons that could be used to prove that a given polygon is regular. Give one illustration of the use of each.

7. Describe briefly the method by which a square may be inscribed in a circle.

8. State the method for circumscribing a square about a circle.

9. State the method for inscribing a regular octagon in a circle and for circumscribing a regular octagon about a circle.

10. State the method for (*a*) inscribing a regular hexagon in a circle; (*b*) circumscribing a regular hexagon about a circle.

11. How would you (*a*) inscribe an equilateral triangle in a circle? (*b*) circumscribe an equilateral triangle about a circle?

12. Answer question 11 for a regular 12-side.

13. What regular polygons can be obtained from the inscribed square and how?

14. Answer question 13 for the regular inscribed hexagon.

15. Quote the theorem concerning the area of a regular polygon.

16. Look through the unit and see what other regular polygons of less than 20 sides can be constructed with ruler and compasses. Make a list of them.

17. What new method of proving polygons similar is given in this unit? Quote in full all the reasons that can be used to prove polygons of more than 3 sides similar.

If, in two regular polygons of the same number of sides r and r' represent the radii, a and a' the apothems, p and p' the perimeters, and A and A' the areas, write the equation connecting the letters in each of the following. Quote the reason in full in each case.

18. A, A', r, and r'

19. r, r', a, and a'

20. r, r', p, and p'

21. A, A', p, and p'

22. p, p', a, and a'

23. A, A', a, and a'

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*
1	368
2	409
3	369
4	370, 371
5	371, 112
6	368, 372
7	374
8	374
9	374
10	375
11	375
12	375
13	374
14	375
15	376
17	378, 357
18	379
19	379
20	379
21	379
22	379
23	379

UNIT X

THE MEASUREMENT OF THE CIRCLE

382. How we get π (pi). In your work in arithmetic you used 3.1416 or $3\frac{1}{7}$ to find the circumference of a circle if you knew its radius. You called this number π. Did you have any idea of where this number came from or why it was used, or did you take it entirely on faith? You are now to learn one of the oldest and most interesting ways of finding the value of π. The exercises on pages 400–405 will show you some of the many practical uses of the measurement of the circle.

383. What is meant by the length of a straight-line segment? How is it found? A straight-line segment cannot be laid on a circle and the measure of the circle found in this way. Some indirect method will have to be used. It will be assumed that the circle can be measured in terms of a straight-line unit. This measure is called the **length of the circle** or its **circumference.**

It is proved in higher mathematics that the circumference of a circle is always an irrational number. What do you know about an irrational number? For practical purposes an approximation to the circumference of the

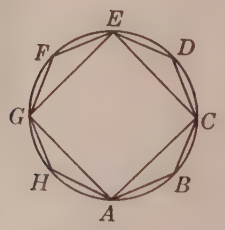


FIG. 556

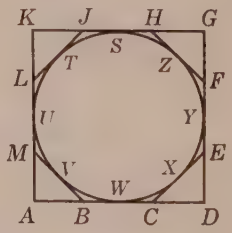


FIG. 557

circle is all that is necessary. Such an approximation is obtained from the perimeters of inscribed and circumscribed polygons of a great many sides.

384. If a square is inscribed in a circle and the arcs between its vertices are bisected, the chords joining the points of division form an inscribed regular octagon (Fig. 556). By the same process, regular polygons of 16, 32, 64, 128 sides may be obtained, and so on indefinitely. If such a series of polygons is constructed, one is soon found which can with difficulty be distinguished from the circle. Instead of the inscribed square, the inscribed regular hexagon might have been used as the starting point.

If a square is circumscribed about a circle and the arcs between the points of tangency are bisected, the tangents drawn at the points of division form a circumscribed regular polygon of 8 sides (Fig. 557). By the same process, circumscribed regular polygons of 16, 32, 64, 128 sides may be obtained, and so on indefinitely. Just as in the series of inscribed regular polygons, so in this case, a polygon is soon found which can with difficulty be distinguished from the circle. Instead of the circumscribed square, the circumscribed regular hexagon might have been the starting point.

385. If a square or a regular hexagon be inscribed in a circle, one side and, therefore, its perimeter may be computed when the radius of the circle is given. From each result the side and, therefore, the perimeter of a polygon of twice as many sides may be computed by a formula given in §415, p. 421. In fact, any regular polygon may be taken as the starting point and the process continued indefinitely. Similarly the perimeters of a series of circumscribed regular polygons may be computed if the radius of the circle is given (see §416, p. 422).

386. The following tables give the results of some of the computations referred to in the previous articles. The radius of the circle is one-half in each case.

TABLE I

Number of Sides	Perimeter of Inscribed Polygon	Perimeter of Circumscribed Polygon
4	2.828427	4.000000
8	3.061667	3.313708
16	3.121445	3.182597
32	3.136548	3.151724
64	3.140331	3.144118
128	3.141277	3.142223
256	3.141513	3.141750
512	3.141572	3.141632
1024	3.141587	3.141602
2048	3.141591	3.141595
4096	3.141592	3.141593
8192	3.141592	3.141592

TABLE II

Number of Sides	Perimeter of Inscribed Polygon	Perimeter of Circumscribed Polygon
6	3.	3.464101
12	3.105828	3.215390
24	3.132628	3.159659
48	3.139350	3.146086
96	3.141031	3.142714
192	3.141452	3.141873
384	3.141557	3.141662
768	3.141583	3.141610
1536	3.141590	3.141597
3072	3.141592	3.141594
6144	3.141592	3.141593

Ex. 1. As the number of sides increases, are the perimeters of the inscribed polygons increasing or decreasing? of the circumscribed polygons?

Ex. 2. Find the difference between the perimeters of the inscribed polygon of 64 sides and the circumscribed polygon of 64 sides; of the corresponding polygons of 128 sides; of 192 sides; of 384 sides; of 512 sides; etc. What can you say of these differences?

Ex. 3. What can you say of the final results in the four columns?

Remember that these numbers really never end and that in these tables they have been cut off to six decimal places only for convenience. It is evident, therefore, that by taking decimal places enough and sides enough to the polygons we can make the differences referred to in the exercises above as small as we please.

387. If you know the perimeter of any regular polygon inscribed in or circumscribed about a circle whose radius is one-half, you can get the perimeters of such polygons when the radius is any number whatsoever by §374, p. 379.

If p is the perimeter of a regular polygon inscribed in or circumscribed about a circle whose radius is one-half, the perimeter of a regular polygon inscribed in or circumscribed about a circle whose radius is r is $2pr$.

388. The circumference of the circle. It is proved in higher mathematics that the perimeters of the inscribed polygons given in Tables I and II can never be greater than a certain definite number. It is also proved that the perimeters of the circumscribed polygons can never be less than that same number. This is true no matter how many decimal places are used or how many sides the polygon may have.

The number referred to in the preceding paragraph has received a definite name. It is called π (*pi*). Note that the radius of the circle in this case is one-half.

389. From §387 and §388 we have:

Post. 47. If series of regular polygons are inscribed in and circumscribed about the same circle (radius r) and the number of sides is increased indefinitely, the limit of the perimeters of these series is the same. This limit is $2\pi r$.

As proved in higher mathematics, this limit does not depend upon the number of sides of the original polygon nor upon the method of increasing the number of sides.

390. From the above postulate is derived the following definition:

The **length or the circumference of a circle** is defined as the limit of the perimeters of a series of regular polygons inscribed in or circumscribed about a circle as the number of sides is increased indefinitely.

391. COR. **The circumference of a circle of radius r is $2\pi r$. $c = 2\pi r = \pi d$.**

π equals the ratio of the circumference of a circle to its diameter.

392. The circumference of a circle can never be expressed exactly in fractions or in decimals. For practical purposes an approximation to π is used.

NOTE. For many centuries approximations to π were found from the perimeters of inscribed and circumscribed polygons as given on page 394. π is very important in all mathematics. It is an irrational number, but very unlike such irrational numbers as $\sqrt{2}$ or $\sqrt{3}$. You found it possible by means of straightedge and compasses to construct a straight-line segment to represent $\sqrt{2}$ (p. 365, Ex. 10; p. 366, Ex. 11). The ancient Greeks could do this. They failed, however, in the attempt to construct a straight-line segment to represent π. In modern times it has been proved that it is impossible to construct such a segment for π by the use of straightedge and compasses. It can, however, be done by means of an instrument called an integraph, which was invented by a Polish mathematician about 1878.

393. The area of the circle. The following statement is proved in higher mathematics. It will be assumed here.

Post. 48. If series of regular polygons are inscribed in and circumscribed about the same circle (radius r) and the number of sides is increased indefinitely, the limit of the areas of these series is the same. This limit is one-half the product of the radius of the circle and its circumference.

394. The **area of a circle** will be defined as the limit of the areas of a series of inscribed or circumscribed regular polygons as the number of sides is increased indefinitely.

395. COR. **The area of a circle is one-half the product of its radius and its circumference.**

396. The formula for the area of the circle. Let A represent the area, r the radius, and c the circumference.

$$A = \tfrac{1}{2}cr \qquad c = 2\pi r \qquad \therefore\ A = \pi r^2$$

397. Areas of sectors of circles. A **sector of a circle** is a figure bounded by two radii and the intercepted arc. The following statement will be assumed.

Post. 49. The area of a sector has the same ratio to the area of a circle of which it is a part as the angle of the sector has to four right angles.

Let A represent the area of the sector.

$$\frac{A}{\pi r^2} = \frac{\text{Angle}}{360}. \quad \text{Solve for } A. \qquad A = \frac{\pi r^2 \text{angle}}{360}.$$

398. Areas of circular segments. A **circular segment** is a figure bounded by an arc and its chord.

From Fig. 558 it is evident that the area of the segment can be found from the area of the sector having the same arc and the area of the associated triangle.

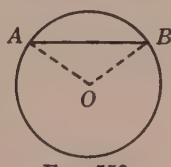


FIG. 558

THEOREM 90

399. The ratio of the circumferences of two circles equals the ratio of their diameters or of their radii.

Hypothesis: c and c' represent the circumferences, r and r' the radii, and d and d' the diameters of any two circles.

Conclusion: $\frac{c}{c'}=\frac{d}{d'}=\frac{r}{r'}$.

Outline of proof. Give the reasons.

1. $c=2\pi r$.
2. $c'=2\pi r'$.
3. $\therefore \frac{c}{c'}=\frac{2\pi r}{2\pi r'}=\frac{2r}{2r'}$.
4. $\therefore \frac{c}{c'}=\frac{d}{d'}=\frac{r}{r'}$.

THEOREM 91

400. The ratio of the areas of two circles equals the ratio of the squares of their radii or of their diameters.

Hypothesis: A and A' represent the areas, r and r' the radii, and d and d' the diameters of any two circles.

Conclusion: $\frac{A}{A'}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$.

Outline of proof. Give the reasons.

1. $A=\pi r^2$.
2. $A'=\pi r'^2$.
3. $\therefore \frac{A}{A'}=\frac{\pi r^2}{\pi r'^2}=\frac{r^2}{r'^2}$.
4. $\frac{r}{r'}=\frac{d}{d'}$.
5. $\therefore \frac{A}{A'}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$.

C Exercises

NOTE. Use $\pi = 3.142$.

1. Find the cirumference of a circle if the diameter is 9 ft.

2. Find the radius of a circle if the circumference is 15 ft.

3. Find the area of a circle if the radius is 25 ft.

4. If the radii of two circles are 18 and 24, what is the ratio of their circumferences? of their areas?

5. If the circumferences of two circles are 24 and 36, what is the ratio of their radii? of their areas?

6. If the areas of two circles are 20 and 27, what is the ratio of their radii? of their circumferences?

7. Construct a circle whose circumference is twice the circumference of a given circle.

8. Construct a circle whose area is twice the area of a given circle.

9. What is the area of a circle inscribed in a square whose side is 6 in.?

10. What is the area of a circle inscribed in a regular hexagon whose side is 6 in.?

11. What is the area of a circle circumscribed about a square whose side is 8 in.?

12. What is the area of a circle circumscribed about a regular hexagon whose side is 12 in.?

13. What is the length of an arc of 40° if the radius of the circle is 20 in.?

14. What is the area of a sector of a circle of radius 14 ft. if the angle of the sector is 72°?

15. If the length of an arc of 24° is 16 ft., find the radius of the circle.

16. Find the area of the segment between one side of an inscribed square and the circle if the radius of the circle is 5 in.

B Exercises

1. What is the diameter of a wheel that makes 250 revolutions in traveling one mile?

2. A locomotive is running at the rate of 40 mi. per hour. If the driving wheels are 80 in. in diameter and the truck wheels are 36 in., how many revolutions per minute does each make?

3. The 72-in. drive wheel of a locomotive makes 250 turns per minute. What is the rate of the locomotive in miles per hour?

4. Two pulleys, one 36 in. and one 15 in. in diameter, are connected by a belt. The 36-in. pulley makes 200 turns per minute. How many turns does the 15-in. pulley make per minute?

5. It is the general practice not to permit the rim of a flywheel to have a greater velocity than a mile a minute. How many revolutions per minute may be allowed an 8-ft. flywheel?

6. Find the length of a circular railway curve of radius one-fourth mile and central angle 30°.

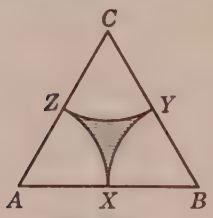


FIG. 559

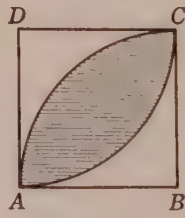


FIG. 560

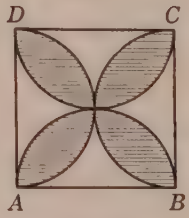


FIG. 561

7. One side of the equilateral triangle in Fig. 559 is 7 ft. Show how the shaded portion is formed and find its area.

8. Show how the shaded figure in the square in Fig. 560 is formed and find its area if one side of the square is 6 in.

9. Work Ex. 8 for Fig. 561.

10. Fig. 562 shows the construction for a certain type of trefoil. Find the entire area if one side of the equilateral triangle is 5 ft.

11. Find the area of a circular segment if its arc is 60°.

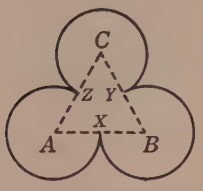


FIG. 562

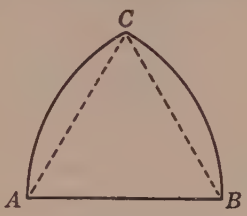


FIG. 563

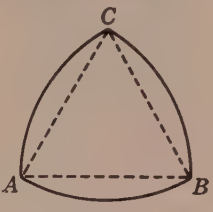


FIG. 564

12. In Fig. 563, the arcs AC and BC are drawn with B and A as centers respectively and AB as radius. $\triangle ABC$ is equilateral. Find the area of the figure if $AB = 6$ ft.

13. Show how Fig. 564 is formed and find its area if $AB = 8$ ft.

NOTE. Figs. 562, 563, and 564 are frequently used in church window designs.

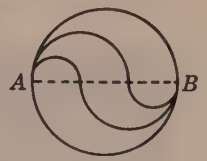


FIG. 565

14. In Fig. 565, the diameter of the circle is divided into three equal parts. Show how the arcs are drawn and prove that they divide the circle into three equal parts.

15. Show how Fig. 566 is formed. Find the area of $OABCD$ if $OB = 5$ ft.

16. Show how to divide a circle into four equal parts by the arcs of circles only.

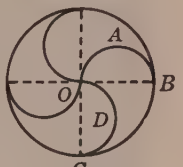


FIG. 566

NOTE. Ex. 16 is said to be a problem which Napoleon once proposed to his staff. Can it be solved in more than one way? Fig. 566 is said to be used by the American Indians to symbolize the four ages of man: infancy, youth, middle, and old age. The figure formed by dividing the circle into two equal parts is the trademark of one of the western railroads. This latter way of dividing the circle is very ancient and appears on charms in China and Chosen (Korea).

17. Show how Fig. 567 is formed and find the area of *HKLMFEH* and of *HKLMFGH* if one side of the square *ABCD* is 6 in.

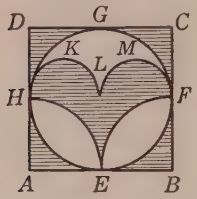


FIG. 567

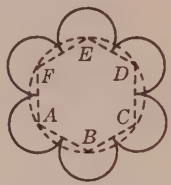


FIG. 568

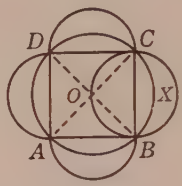


FIG. 569

NOTE. Fig. 567 is from an old Roman pavement. It is frequently seen in ornamental iron grilles.

18. Show how Fig. 568 is formed and find its area if a side of the hexagon is 5 ft.

19. Show how Fig. 569 is formed and find the entire area inclosed if $AB = 8$ in.

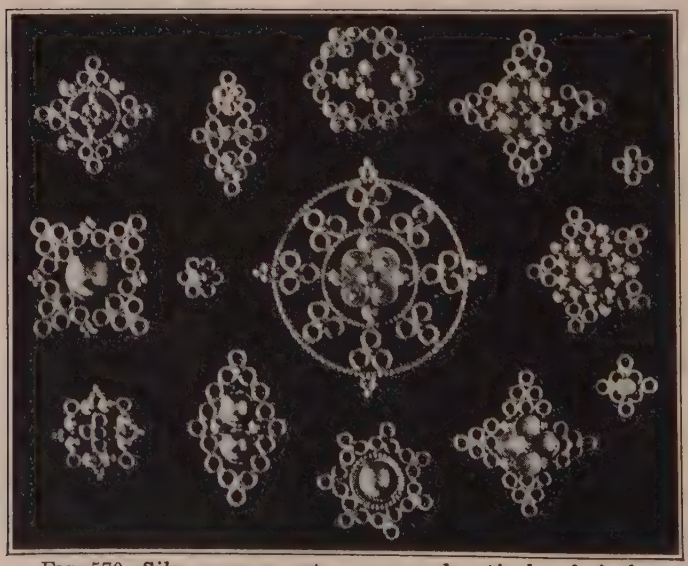


FIG. 570. **Silver ornaments composed entirely of circles**

20. In Fig. 569, show how the curved figure $BXCO$ is formed and find its area if $BC = 10$ in.

21. In Fig. 571, D and X are the mid-points of AO and OB respectively. $AO = OB$. Prove that the area bounded by the four semicircles equals the area of the circle on CF as diameter.

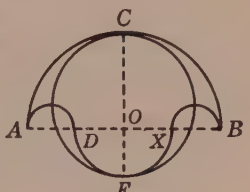


FIG. 571

NOTE. Archimedes solved this problem. He called it a *salenon* (salt cellar).

A Exercises

1. If one side of a square is b, show that the area of the circumscribed circle is $\frac{1}{2}\pi b^2$.

2. If the sum of the radii of two given circles is equal to the radius of a third circle, prove that the circumference of the third circle is equal to the sum of the circumferences of the two given circles.

3. Find a formula for the area of the ring between concentric circles if the radii are r and r' respectively.

4. Fig. 572 shows two concentric circles. Show how to find the segment AB so that the circle constructed on AB as diameter shall have the same area as the ring.

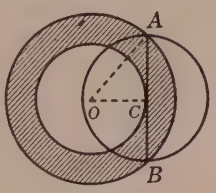


FIG. 572

5. In Fig. 573, C is the mid-point of radius AO of $\odot O$. XC is perpendicular to AO at C. Prove that the circle with O as center and OX as radius has half the area of the circle with radius AO.

6. Show how to divide a circle into three equal parts by concentric circles.

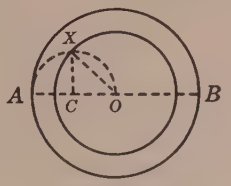


FIG. 573

7. If a circle is constructed on each of the sides of a right triangle as diameter, the area of the circle on the hypotenuse is equal to the sum of the areas of the other two circles.

8. Find by geometrical construction the diameter of a pipe the area of whose cross section is equal to the sum of the areas of the cross sections of two given pipes.

9. A 12-in. water pipe branches into three equal pipes whose combined capacity is the same as that of the 12-in. pipe. If the quantity of water carried depends upon the area of the cross section of the pipes, what must be the diameter of each of the three pipes?

10. Semicircles are constructed on the sides of a right triangle as shown in Fig. 574. Show that the sum of the areas of the two crescents is equal to the area of the triangle.

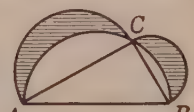


FIG. 574

NOTE. This problem is due to Hippocrates (about 470 B.C.). His solution is the first case of the area of a curvilinear figure proved equal to the area of a rectilinear figure.

11. The rail of a street-car track is 12 ft. from the curb. In rounding a square corner it cannot be closer than 3 ft. from the curb. What is the radius of curvature of the required curve? How far back must the curve begin? (Fig. 575.)

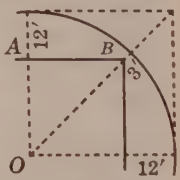


FIG. 575

12. A circular grass plot 16 ft. in diameter is cut by a straight gravel path 4 ft. wide, one edge of which passes through the center of the plot. What is the area of the remaining grass plot?

13. It is desired to construct a half-mile track. The start and finish are to be straightways intersecting at right angles at the goal. The rest of the track is to be an arc of a circle tangent to the two straightways. Find the radius of the arc and the length of the arc in feet. Result correct to two decimal places.

14. The cross section of the sewer in Fig. 576 is made up of four circular arcs AB, BC, CD, and DA. The center of arc AB is Y; of arc AD is P; of arc DC is X. The figure is symmetric with respect to line XY. If $YB = 1.2$ ft., $XE = .7$ ft., $XY = 2.0$ ft., $\angle P = 30°$, find the entire circumference of the inner line of the sewer. AB is a semicircle. (See p. 161, *optional.*)

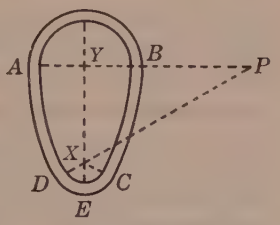


FIG. 576

A Exercises — Approximations for π

We have seen that numbers can be found which more or less closely approximate the value of π, although a number cannot be found which is exactly equal to π. Similarly, segments can be constructed whose lengths more or less closely approximate the value of π, although no segment can be constructed with ruler and compasses whose length exactly represents the value of π. The following exercises give some of these constructions. They are used by draftsmen in obtaining the development of cylinders, by carpenters in obtaining the dimensions of veneers for semicircular heads of doors and windows, and by mechanical tradesmen generally. The computations for the true lengths of the segments often involve considerable geometry, and are necessary in order to ascertain the degree of approximation obtained.

1. Determine the value of π by the following experiment: Wrap a paper about a cylinder. Stick a pin through the overlapped paper. Measure the distance between the two pinholes and the diameter of the cylinder and compute π.

2. In Fig. 577, ACB is a semicircle with AB its diameter. $OC \perp$ bisector of AB. $\angle OCD = 30°$. Find the difference between the length of AD and of the arc AC, using $AB = 5$. Find the ratio of this difference to the length of the arc AC.

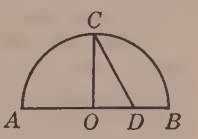


FIG. 577

3. In Fig. 578, ACB is a semicircle with AB its diameter. $\triangle ABD$ is equilateral, with AB as one side. EF is a tangent to the semicircle parallel to AB. DA and DB are extended to meet the tangent EF at F and E respectively. Using $AB=6$, find the difference between the length of EF and of the semicircle ACB. Find the ratio of this difference to the length of ACB.

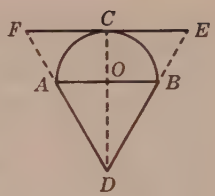


FIG. 578

4. In Fig 579, $\angle AOE$ is a central angle of 45° in circle O. $OE \perp AB$. Using $OE=8$, find the length of DE and the difference between $6OE+DE$ and the circumference. Find the ratio of this difference to the circumference.

5. Inscribe a square in a given circle. Find the value of three times the diameter of the circle plus one-fifth of the side of the square, using the diameter of the circle as d. Find the difference between this sum and the circumference of the circle. Find the ratio of this difference to the circumference.

FIG. 579

NOTE. It is not known who made the first attempt to compare the area of a circle with the area of a square. But the first record that has so far been found is on an Egyptian papyrus by Ahmes (about 1700 B.C.). He says, "Cut off $\frac{1}{9}$ of a diameter and construct a square on the remainder." This implies that the area of the circle is $(\frac{8}{9}d)^2$ or $\frac{256}{81}r^2$; that is, the approximation to π is $\frac{256}{81}=3.16049+$. This is a much better approximation than was used by some other ancient peoples. In the countries of Asia 3 was commonly used. Both 3 and $3\frac{1}{2}$ are found in the Bible (I Kings 7:23; Daniel 7:25). Archimedes (287–212 B.C.) used inscribed and circumscribed polygons of 96 sides and showed that π was less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. His method was practically the only method used for the next 2,000 years until the invention of the calculus in modern times. $3\frac{1}{7}$ is an approximation close enough for most practical work. About 150 A.D. Ptolemy gave a value of 38′ 30″ (equivalent to 3.1416). During the Middle Ages, before the invention of decimals, various approximations were used in Europe and among the Hindus and Arabs. Among them were

$3\frac{1}{7}$, $\sqrt{10}$, $3\frac{177}{1250}$, $\frac{62832}{20000}$. $\sqrt{10}$ may have been used for that purpose because of a very early approximation of $\sqrt{10}$: $\sqrt{10} = \sqrt{9+1} = 3+\frac{1}{2\times 3+1} = 3\frac{1}{7}$. About the beginning of the seventeenth century a European published a result correct to 35 decimal places. A hundred years or so later, new methods of computation by means of never-ending series were devised. In 1874 a value was found showing 707 decimal places. The first 101 digits are as follows: $\pi = 3.1415926535$ 89793238462643383279502884197169399375105820974944592307816406 28620899862803482534211706798+

Survey Test

1. What do you understand by the length of a circle?

2. Why is an approximation to the circumference of a circle used? Where do we get such an approximation?

3. If a series of regular polygons is inscribed in a circle of radius r and the number of sides is increased indefinitely, how does the length of each side change? How does the radius change? the apothem? the perimeter? the area? Is there any limit in each case, and if so what?

4. Answer question 3 for series of regular polygons circumscribed about a circle.

5. If p is the perimeter of a regular polygon inscribed in or circumscribed about a circle whose radius is one-half, what is the perimeter of a regular polygon inscribed in or circumscribed about a circle whose radius is r?

6. What is π? Explain clearly how approximations to the value of π may be obtained.

7. To how many decimal places has π been computed? Why are only six given on page 394?

8. (*a*) Define the circumference of a circle. (*b*) Give the formula for finding it.

9. (*a*) Define the area of a circle. (*b*) Give the formula for finding it.

10. Are the answers obtained from the formulas given in answers 8 and 9 exact? Why?

11. Define a sector of a circle. Give the formula for finding its area. Quote the postulate from which this formula was obtained.

12. What data must be given before you can find the area of a sector?

13. Define a segment of a circle. How can you find its area?

14. What data would be necessary before you can find the area of a segment of a circle? Is the problem always possible without trigonometry? Why?

If in two circles c and c' represent the circumferences, A and A' the areas, and r and r' the radii, write the equations connecting the letters in each of the following. Quote the reason in full in each case.

15. A, A, r, and r'.

16. c, c', r, and r'.

17. A, A, c, and c'.

Remedial Study

If you are unable to answer any of the foregoing questions, consult the following pages for help:

Question	*Page*	*Question*	*Page*	*Question*	*Page*
1	396	6	396, 393–394	11	397
2	392	7	407	12	397
3	396, 397	8	396	13	397
4	396, 397	9	397	15	398
5	395			16	398

401. SUMMARY IV

Regular Polygons and Circles

REGULAR POLYGONS

A polygon is regular if:

1. The sides and angles are equal ..... §351, p. 368
2. A circle is divided into equal arcs and
   a. The points of division are joined ..... Th. 86, p. 372
   b. Tangents are drawn to the points of division ... Th. 86, p. 372

Properties of regular polygons:

1. The sides and angles are equal ..... §351, p. 368
2. A circle can be circumscribed about the polygon ... Th. 84, p. 369
3. A circle can be inscribed in the polygon ..... Th. 85, p. 370
4. The radius bisects the angle ..... §354, p. 369
5. The apothem is perpendicular to the sides ..... §358, p. 371
6. Each angle is $\frac{2n-4}{n}$ rt. $\angle$ ..... §132, p. 112

Construction of regular polygons:

1. For inscribed or circumscribed regular polygons of n sides, divide the circle into n equal arcs by constructing an angle of $\frac{1}{n}$ 360°, at the center of the circle ..... Th. 86, p. 372
2. For 4-, 8-, or 16-sided regular polygons, construct two perpendicular diameters ..... Prob. 20, §§364–365, p. 374
3. For 3-, 6-, or 12-sided regular polygons, construct a central angle of 60° by means of an equilateral triangle ..... Prob. 21, §§367–369, p. 375

‡4. For 5-, 10-, or 15-sided regular polygons, divide the radius of the circle into extreme and mean ratio Prob. 23, p. 386; §§379–380, p. 387

The area of a regular polygon is ½ the product of the perimeter and the apothem ..... Th. 87, p. 376

‡Optional.

Similar regular polygons:

1. Regular polygons are similar if they have the same number of sides. . . . . Th. 88, p. 378
2. The ratio of the radii or of the apothem equals the ratio of two corresponding sides. . . . . Th. 89, p. 379
3. The ratio of the perimeters equals the ratio of the radii or of the apothems. . . . . §373, p. 379
4. The ratio of the areas equals the ratio of the squares of the radii or of the apothems. . . . . §375, p. 379

CIRCLES

Formulas for circles:

1. For one circle
   $c = 2\pi r$ . . . . . §391, p. 396
   $A = \pi r^2$ . . . . . §396, p. 397
2. For two circles
   $\frac{c}{c'} = \frac{d}{d'} = \frac{r}{r'}$ . . . . . Th. 90, p. 398
   $\frac{A}{A'} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}$ . . . . . Th. 91, p. 398
3. For sectors
   $A = \frac{\pi r^2 \text{ angle}}{360}$ . . . . . §397, p. 397

C Exercises

1. What regular polygons of less than 100 sides can be constructed? Make a list and arrange in numerical order.

2. What regular polygon has a perimeter equal to eight times its apothem?

3. What regular polygon has its side equal to its radius?

4. What is the ratio of the circumference of a circle circumscribed about a square to that of the circle inscribed in the same square?

5. What is the ratio of the areas of the two circles mentioned in Ex. 4?

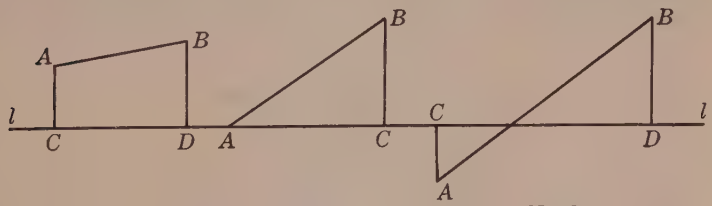
6. Show how to find the area of a segment of a circle if its arc is more than 180°.

SUPPLEMENT

I. Some Formulas

402. The theorems and problems given on pp. 412–423 are not in the list of fundamental propositions recommended by the National Committee on Mathematical Requirements, nor are they in the list of the College Entrance Examination Board. Some syllabi, such as that of New York state, require that pupils understand the meaning of these formulas and be able to use them if they are supplied to them as needed. The formulas in §413, §414, and §416 are not in the New York state list.

403. Projection. By the **projection of a point on a line** is meant the foot of the perpendicular from the point to the line; thus, in Fig. 580, Nos. 1 and 3, C is the projection of A on line l if $AC \perp l$.



No. 1 No. 2 No. 3

Fig. 580

By the **projection of a given segment on a given line** is meant the segment of the given line between the projections of its end points; thus, in Fig. 580, CD is the projection of AB on line l. It will be written $CD = p_l^{AB}$. Note that, when one end of the given segment is on the given line, it is necessary to project the other end only on the given line; thus, in Fig. 580, No. 2, $AC = p_l^{AB}$.

THEOREM 92

404. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other side upon it.

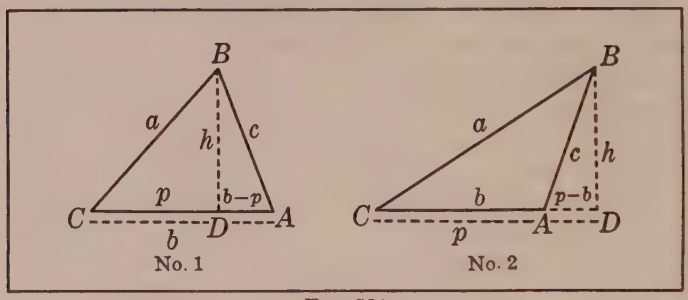


FIG. 581

Hypothesis: In $\triangle ABC$, $\angle C$ is acute. $CD = p_b^a$.

Conclusion: $c^2 = a^2 + b^2 - 2bp_b^a$.

Plan: Use the Pythagorean Theorem and eliminate h.

Outline of proof. Give the reasons (Fig. 581, No. 1).

1. $c^2 = (b-p)^2 + h^2$.
2. $= b^2 - 2bp + p^2 + h^2$.
3. $= b^2 - 2bp + a^2$.
4. $= b^2 + a^2 - 2bp_b^a$.

Exercise. Give the proof for Fig. 581, No. 2.

405. COR. $\boldsymbol{p_b^a = \dfrac{a^2 + b^2 - c^2}{2b}}$.

Ex. 1. Prove that the projection of either leg of an isosceles triangle on the base is equal to half the base.

Ex. 2. Write the formula for a^2 and for b^2 when all the angles of the triangle are acute.

Ex. 3. If $a = 30$, $b = 24$, and $c = 25$, find p_b^a.

Ex. 4. If $b = 24$ and $c = 25$, what would a be if $p = 0$?

THEOREM 93

406. In any triangle having an obtuse angle, the square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other side upon it.

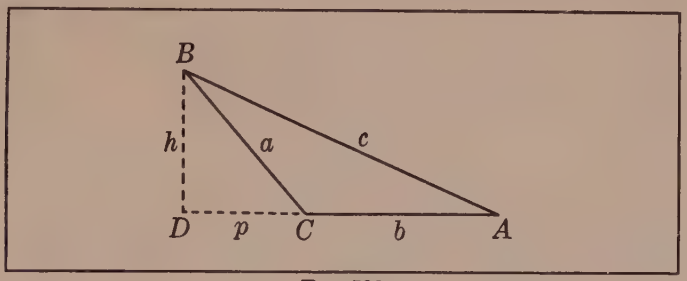


FIG. 582

Hypothesis: In $\triangle ABC$, $\angle C$ is obtuse. $CD = p_b^a$.

Conclusion: $c^2 = a^2 + b^2 + 2bp_b^a$.

Plan: Use the Pythagorean Theorem and eliminate h.

Outline of proof. Give the reasons.

1. $c^2 = (b+p)^2 + h^2$.
   $= b^2 + 2bp + p^2 + h^2$.
   $= b^2 + 2bp + a^2$.
   $= b^2 + a^2 + 2bp_b^a$.

407. COR. $\boldsymbol{p_b^a = \dfrac{c^2 - b^2 - a^2}{2b}}$.

408. If a, b, and c are the sides of a triangle and if

$\angle c$ is acute, $c^2 < a^2 + b^2$
$\angle c$ is right, $c^2 = a^2 + b^2$
$\angle c$ is obtuse, $c^2 > a^2 + b^2$

Exercise. Is the largest angle of a triangle acute, right, or obtuse (*a*) if $a = 15$, $b = 8$, and $c = 20$? (*b*) if $a = 15$, $b = 8$, and $c = 15$?

PROBLEM 24

409. To find the area of a triangle in terms of the sides.

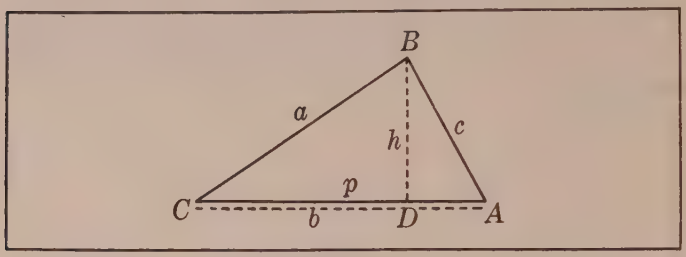


FIG. 583

Given ABC any triangle with the sides a, b, and c.

To find a formula for the area of the triangle in terms of a, b, and c.

Since every triangle has at least two acute angles, assume that $\angle C$ is acute. Construct BD the altitude on side b. Then $CD = p_b^a$. Let A represent the area of the triangle.

Outline of work. Give all reasons.

1. $A = \frac{1}{2}hb$.
2. $A^2 = \frac{1}{4}h^2b^2 = \frac{b^2}{4}h^2$.
3. $= \frac{b^2}{4}(a^2 - p^2) = \frac{b^2}{4}(a+p)(a-p)$.
4. $= \frac{b^2}{4}\left(a + \frac{a^2+b^2-c^2}{2b}\right)\left(a - \frac{a^2+b^2-c^2}{2b}\right)$ .... see §405.
5. $= \frac{b^2}{4}\left(\frac{2ab+a^2+b^2-c^2}{2b}\right)\left(\frac{2ab-a^2-b^2+c^2}{2b}\right)$.
6. $= \frac{(a^2+2ab+b^2-c^2)(c^2-a^2+2ab-b^2)}{16}$.

7. $A^2 = \dfrac{[(a+b)^2 - c^2]\,[c^2 - (a-b)^2]}{16}$.

8. $= \dfrac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{16}$.

9. $= \dfrac{(a+b+c)(a+b+c-2c)(a+b+c-2b)(a+b+c-2a)}{16}$.

10. Let $a+b+c=2s$.

11. $\therefore A^2 = \dfrac{2s(2s-2c)(2s-2b)(2s-2a)}{16}$.

12. $= s(s-c)(s-b)(s-a)$.

13. $\therefore A = \sqrt{s(s-c)(s-b)(s-a)}$.

The same result would be obtained if the formula for p were taken from §407.

410. COR. **To find the altitude of a triangle on side a:**

$$A = \tfrac{1}{2} h_a A.$$

$$\therefore h_a = \frac{2A}{a} = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 1. Find the area of a triangle whose sides are 13, 20, and 21.

Ex. 2. Find by §410 the altitude upon each of the sides of the triangle mentioned in Ex. 1.

Ex. 3. Find the area of a triangle whose sides are 15, 17, and 8.

Ex. 4. Find the altitude of the triangle mentioned in Ex. 3 on (*a*) side 17; (*b*) side 8.

Ex. 5. Is the angle opposite side 17 in the triangle mentioned in Ex. 4 acute, right, or obtuse? Why?

Ex. 6. Find by §409 and §410 the altitude and the area of an equilateral triangle each of whose sides is 12.

Ex. 7. Answer Ex. 6 for an equilateral triangle each of whose sides is b.

Problem 25

411. To find in any triangle the length of the bisector of angle A.

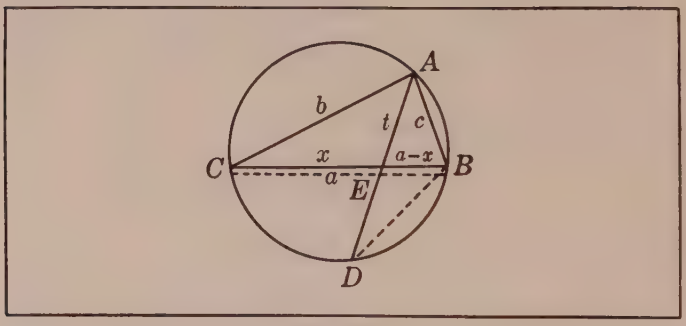


Fig. 584

Given $\triangle ABC$ with AE bisecting $\angle A$. Let $AE = t_A$.

To find the length of AE or t_A.

Circumscribe a circle about $\triangle ABC$. Continue AE to meet the circle at D. Draw BD.

Outline of work. Give all details and reasons.

1. $\triangle ABD \sim \triangle AEC$.
2. $\therefore \dfrac{t+ED}{b} = \dfrac{c}{t}$.
3. $\therefore t^2 + t(ED) = bc$.
4. $t(ED) = x(a-x)$ . . . . . Th. 66, p. 276
5. $\dfrac{x}{a-x} = \dfrac{b}{c}$ . . . . . Th. 63, p. 269
6. $\dfrac{a}{x} = \dfrac{b+c}{b}$ . . . . from 5 by §268, p. 256
7. $\therefore x = \dfrac{ab}{b+c}$.
8. $a - x = a - \dfrac{ab}{b+c} = \dfrac{ac}{b+c}$.

9. $t^2+x(a-x)=bc$ . . . . . from 3 and 4

10. $\therefore t^2+\frac{ab}{b+c}\cdot\frac{ac}{b+c}=bc.$

11. $\therefore t^2=bc-\frac{a^2bc}{(b+c)^2}.$

12. $=bc\left[1-\frac{a^2}{(b+c)^2}\right].$

13. $=bc\left[\frac{(b+c)^2-a^2}{(b+c)^2}\right].$

14. $=bc\left[\frac{(b+c+a)(b+c-a)}{(b+c)^2}\right].$

15. Let $2s=a+b+c.$

16. Then $2(s-a)=b+c-a.$

17. Then $t^2=bc\left[\frac{2s\cdot 2(s-a)}{(b+c)^2}\right].$

$$t^2=\frac{4bcs(s-a)}{(b+c)^2}.$$

18. $$\therefore t_A=\frac{2}{b+c}\sqrt{bcs(s-a)}.$$

The sides of a triangle are 10, 17, and 21.

Ex. 1. Is the largest angle acute, right, or obtuse?

Ex. 2. Find the lengths of the bisectors of the three angles.

Ex. 3. Find the lengths of the three altitudes.

Ex. 4. Find the area by §409, p. 414.

Ex. 5. After you have worked Exs. 3 and 4, check the results by finding the area by the formulae

$$A=\tfrac{1}{2}h_aa=\tfrac{1}{2}h_bb=\tfrac{1}{2}h_cc.$$

Ex. 6. Work Exs. 1–5 for a triangle whose sides are 13, 14, and 15.

Ex. 7. Work Exs. 1–5 for a triangle whose sides are 7, 24, and 25.

Ex. 8. Work Exs. 1–5 for an isosceles triangle whose base is 18 and one of whose legs is 15.

PROBLEM 26

412. To find in any triangle the length of median on side a.

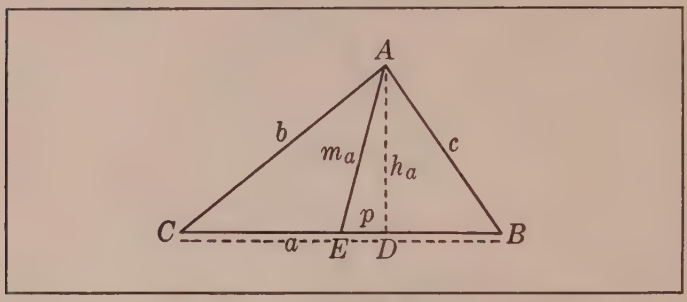


FIG. 585

Given ABC is any triangle. Let $\angle C$ be an acute $\angle$. AE = the median on side a.

To find m_a in terms of the sides.

Draw AD the altitude on side a; then $ED = p_a^{EA}$. Let $\angle CEA$ be obtuse and $\angle BEA$ acute.

Outline of work. Give the reasons.

1. $b^2 = m^2 + \left(\frac{a}{2}\right)^2 + 2\frac{a}{2}p.$
2. $c^2 = m^2 + \left(\frac{a}{2}\right)^2 - 2\frac{a}{2}p.$
3. $b^2 + c^2 = 2m^2 + \frac{a^2}{2}.$
4. $2m^2 = b^2 + c^2 - \frac{a^2}{2}.$
5. $m^2 = \frac{2b^2 + 2c^2 - a^2}{4}.$
6. $\therefore\ m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$

Exercise. Find the length of the medians of a triangle whose sides are 10, 12, and 15.

PROBLEM 27

413. To find the radius of a circle circumscribed about a triangle.

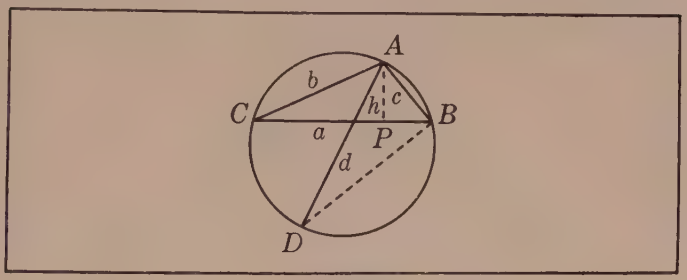


FIG. 586

Given $\triangle ABC$ any triangle and its circumscribed circle whose diameter $AD = d$. Let r be the radius of the circle.

To find r in terms of the sides of the triangle.

Draw AP the altitude on side a. Join B and D.

Outline of work. Give all details and reasons.

1. $\triangle ADB \sim \triangle ACP$.
2. $\therefore \frac{d}{b} = \frac{c}{h}$.
3. $\therefore d = \frac{bc}{h}$.
4. $\therefore r = \frac{bc}{2h}$.
5. Since $A = \frac{1}{2}ah$,
6. $h = \frac{2A}{a}$.
7. $\therefore r = \frac{abc}{4A} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

Problem 28

414. To find the radius of the circle inscribed in any triangle.

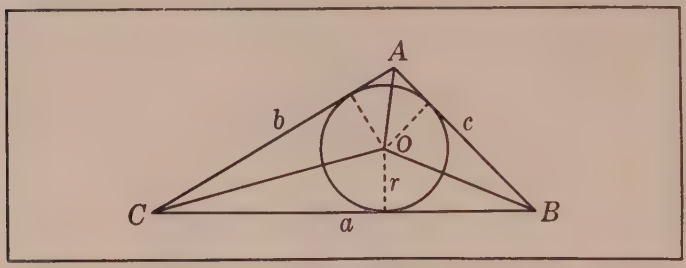


Fig. 587

Given $\triangle ABC$ any triangle and its inscribed circle with radius r. OA, OB, and OC are drawn from the center O to the vertices of the triangle.

To find r in terms of a, b, and c.

Draw the radii to the points of contact.

Outline of work. Give all reasons.

1. $A = \frac{1}{2}rc + \frac{1}{2}rb + \frac{1}{2}rc.$
2. $= r\dfrac{a+b+c}{2}.$
3. $= rs$ where $2s = a+b+c.$
4. $\therefore r = \dfrac{A}{s}.$
5. $A = \sqrt{s(s-a)(s-c)(s-b)}.$
6. $\therefore r = \sqrt{\dfrac{s(s-a)(s-c)(s-b)}{s^2}}.$
7. $= \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}.$

Problem 29

415. To find a side of a regular inscribed polygon of $2n$ sides, given a side of a regular inscribed polygon of n sides and the radius of the circle.

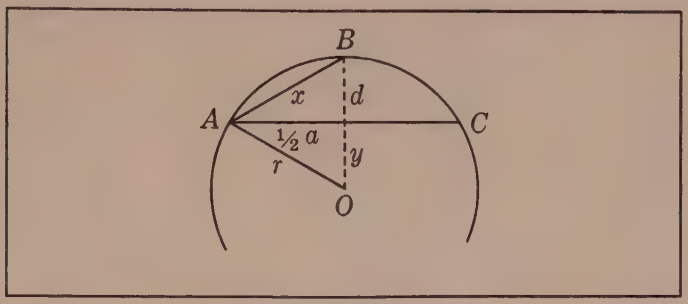


Fig. 588

Given a side AC (or a) of a regular polygon of n sides inscribed in a circle of radius r.

Required to find a side x of an inscribed regular polygon of $2n$ sides.

Analysis (segments are lettered as shown in Fig. 588):

1. To find x in terms of the known lengths, use $x^2 = d^2 + (\frac{1}{2}a)^2$. a is known; d must be found.
2. To find d, use $d = r - y$. $\begin{cases} r \text{ is known.} \\ y \text{ must be found.} \end{cases}$
3. To find y, use $y^2 = r^2 - (\frac{1}{2}a)^2$. a and r are known.

Outline of solution. Give the reasons.

1. $y^2 = r^2 - (\frac{1}{2}a)^2$.
2. $d = r - y = r - \sqrt{r^2 - \frac{1}{4}a^2}$.
3. $x^2 = d^2 + (\frac{1}{2}a)^2 = (r - \sqrt{r^2 - \frac{1}{4}a^2})^2 + \frac{1}{4}a^2$.
   $= r^2 + r^2 - \frac{1}{4}a^2 - 2r\sqrt{r^2 - \frac{1}{4}a^2} + \frac{1}{4}a^2$.
   $= 2r^2 - 2r\sqrt{r^2 - \frac{1}{4}a^2}$.
4. $\therefore x = \sqrt{2r^2 - r\sqrt{4r^2 - a^2}}$.

Problem 30

416. To find a side of a circumscribed regular polygon of $2n$ sides, given a side of a circumscribed regular polygon of n sides and the radius of the circle.

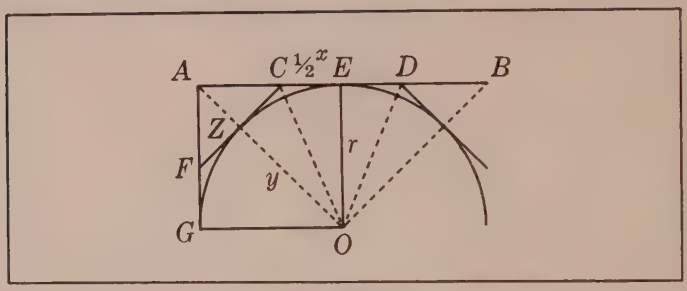


Fig. 589

Given a side AB (or a) of a regular polygon of n sides circumscribed about a circle of radius r.

Required to find a side x of a circumscribed regular polygon of $2n$ sides.

Analysis:

1. To find x, use Th. 63, p. 269.

$$\frac{r}{y} = \frac{\frac{1}{2}x}{\frac{1}{2}a - \frac{1}{2}x}.$$

a and r are known. y must be found.

2. To find y, use $y^2 = (\frac{1}{2}a)^2 + r^2$.
3. To use Th. 63, it is necessary to prove CO bisects $\angle AOE$.

Outline of solution. Give the reasons.

1. To prove CO bisects $\angle AOE$.
   - *a.* AO bisects $\angle EOG$.
   - *b.* $\therefore AO$ bisects $\overset{\frown}{EG}$.
   - *c.* Z is mid-point of $\overset{\frown}{EG}$ (Th. 86, p. 372).
   - *d.* $\therefore AO$ passes through Z.

e. $\therefore CO$ is a radius and ZO and EO are apothems of the circumscribed polygon of $2n$ sides.

f. $\therefore CO$ bisects $\angle ZOE$ ($\angle AOE$). See Ex. 2, p. 371.

2. To find x:

a. $$\frac{r}{y} = \frac{\frac{1}{2}x}{\frac{1}{2}a - \frac{1}{2}x}.$$

b. $$\frac{r}{y} = \frac{x}{a - x}.$$

c. $$xy = ar - rx.$$

d. $$rx + xy = ar.$$

e. $$y^2 = r^2 + (\tfrac{1}{2}a)^2.$$

f. $$rx + x\sqrt{r^2 + \tfrac{1}{4}a^2} = ar.$$

g. $$rx + \tfrac{1}{2}x\sqrt{4r^2 + a^2} = ar.$$

h. $$2rx + x\sqrt{4r^2 + a^2} = 2ar.$$

i. $$\therefore x = \frac{2ar}{2r + \sqrt{4r^2 + a^2}}.$$

Ex. 1. Show that if the radius of a circle is $\frac{1}{2}$, the side of the regular inscribed 12-side is $\frac{1}{2}\sqrt{2 - \sqrt{3}}$.

Ex. 2. From the result obtained in Ex. 1 find the perimeter of the regular inscribed 12-side correct to 2 decimal places.

Ex. 3. Show that if the radius of a circle is $\frac{1}{2}$, the side of the regular inscribed octagon is $\frac{1}{2}\sqrt{2 - \sqrt{2}}$.

Ex. 4. From the result obtained in Ex. 3 find the perimeter of the regular inscribed octagon correct to 2 decimal places.

Ex. 5. Show that if the radius of a circle is $\frac{1}{2}$, the side of the regular circumscribed octagon is $\sqrt{2} - 1$.

Ex. 6. From the result obtained in Ex. 5 find the perimeter of the regular inscribed octagon correct to 2 decimal places.

Ex. 7. If the sides of a triangle are 3, 4, and 5, find the radius of the inscribed circle and of the circumscribed circle.

II. Notes on Arithmetic and Algebra

417. The rule for square root is based on the algebraic formula $(a+b)^2=a^2+2ab+b^2$. Notice that $a^2+2ab+b^2$ may be written $a^2+b(2a+b)$. The method is illustrated below:

Illustration 1. Find $\sqrt{694.563}$

	6,94.56,3 | 26.3
	4
2(20) = 40	294
40 + 6 = 46	276
2(260) = 520	1856
520 + 3 = 523	1569
	287

In the illustration above, notice:

(1) The number was divided into periods of two figures each, counting to the left and to the right from the decimal point.

(2) The largest square under 6 is 4. The 4 was subtracted from 6 and the next period annexed. This gave a remainder of 294. The square root of 4, or 2, was written as the first figure in the root.

(3) A zero was placed after the 2, making 20. The 20 was doubled, making 40. The 40 is used as a trial divisor for the remainder 294. The next figure of the root is either 6 or 7. The 6 is added to the 40, making 46. The 46 is multiplied by 6, giving 276. The 276 is subtracted from 294, leaving 18. The next period is annexed, giving 1856.

(4) The process above is repeated at each step of the work; thus, a zero is placed after 26 and the result doubled, giving 520. The work is then continued as above. In general we may say: Annex a zero to the part of the

root already found and double the result. To this result add the next figure of the root. Multiply the result by the last figure of the root found.

Show that this statement may be regarded as a translation of $b(2a+b)$ in the formula $(a+b)^2=a^2+b(2a+b)$.

Find the square root of the following:

1. 1369	4. 106276	7. 5
2. 3744	5. 3	8. 15
3. 2304	6. 6	9. 7

Because of the frequent occurrence of the square roots of 2 and 3 in geometry work, the application of the following law should be noted:

418. The square root of a product is the product of the square roots of the factors.

Illustration 1. $36=4\times9 \quad \therefore \sqrt{36}=\sqrt{4}\times\sqrt{9}$

This law is used most conveniently for inexact square roots when one factor is a perfect square.

Illustration 2. $18=9\times2 \quad \therefore \sqrt{18}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$

Notice that $\sqrt{2}$ occurs when the side of a square and a diagonal of the square are used in the same exercise.

Illustration 3. $12=4\times3 \quad \therefore \sqrt{12}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$

Notice that $\sqrt{3}$ occurs when the side of an equilateral triangle and its altitude occur in the same exercise.

Illustration 4. $20=4\times5 \quad \therefore \sqrt{20}=\sqrt{4}\times\sqrt{5}=2\sqrt{5}$

The $\sqrt{5}$ occurs in connection with the regular decagon and pentagon.

Find the value of the following correct to three decimal places. Apply the law given above.

1. $\sqrt{8}$	4. $\sqrt{108}$	7. $\sqrt{128}$	10. $\sqrt{150}$	13. $\sqrt{54}$
2. $\sqrt{75}$	5. $\sqrt{32}$	8. $\sqrt{147}$	11. $\sqrt{125}$	14. $\sqrt{45}$
3. $\sqrt{27}$	6. $\sqrt{80}$	9. $\sqrt{320}$	12. $\sqrt{98}$	15. $\sqrt{180}$

16. $\sqrt{200}$ 18. $\sqrt{48}$ 20. $\sqrt{50}$ 22. $\sqrt{288}$ 24. $\sqrt{243}$

17. $\sqrt{96}$ 19. $\sqrt{72}$ 21. $\sqrt{300}$ 23. $\sqrt{162}$ 25. $\sqrt{242}$

419. The square root of a fraction.

A. **If the denominator is a perfect square:** Find the square root of the numerator and of the denominator separately and divide the first result by the second.

Illustration 1. $\sqrt{\frac{144}{625}} = \frac{12}{25}$

Illustration 2. $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$

B. **If the denominator is not a perfect square:** Two methods are suggested:

(1) The fraction may be reduced to a decimal and the square root of the result found.

(2) Numerator and denominator may be multiplied by some number that will make the denominator a perfect square, and the first method, A, above used.

Illustration 3. To find $\sqrt{\frac{1}{3}}$, either

1. Find square root of .333333+; or
2. Write $\frac{1}{3} = \frac{3}{9}$ and use $\frac{1}{3}\sqrt{3}$.

Find the value of the following correct to three decimal places:

1. $\sqrt{\frac{7}{25}}$ 5. $\sqrt{\frac{2}{5}}$ 9. $\sqrt{\frac{4}{5}}$ 13. $\sqrt{\frac{5}{6}}$

2. $\sqrt{\frac{11}{16}}$ 6. $\sqrt{\frac{3}{7}}$ 10. $\sqrt{\frac{9}{11}}$ 14. $\sqrt{\frac{9}{20}}$

3. $\sqrt{\frac{2}{9}}$ 7. $\sqrt{\frac{5}{8}}$ 11. $\sqrt{\frac{7}{18}}$ 15. $\sqrt{\frac{3}{8}}$

4. $\sqrt{\frac{1}{5}}$ 8. $\sqrt{\frac{5}{12}}$ 12. $\sqrt{\frac{2}{3}}$ 16. $\sqrt{\frac{7}{27}}$

420. Fractions. The following fundamental law of fractions underlies all operations that involve fractions:

Multiplying or dividing numerator and denominator of a fraction by the same number does not alter the value of the fraction.

A. **The sum or difference of two or more fractions** that have a common denominator is the sum or difference of the numerators divided by the common denominator.

Two or more fractions that have not a common denominator must be reduced to a common denominator before adding or subtracting. To reduce fractions to a common denominator, apply the fundamental law given above.

Add and subtract the following:

1. $\frac{5}{12}+\frac{7}{18}$
2. $\frac{11}{24}+\frac{13}{36}-\frac{5}{6}$
3. $\frac{a}{b}+\frac{b}{c}$
4. $\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}$
5. $4-\frac{x}{y}+\frac{4x^2}{4y^2}$
6. $\frac{x}{x+1}-\frac{1}{x-1}$

B. **The product of two fractions** is the product of the numerators divided by the product of the denominators.

Where possible, divide numerator and denominator by common factors.

Multiply the following:

1. $\frac{3}{16}\times\frac{8}{15}$
2. $\frac{3a}{b}\times\frac{b^2}{c}$
3. $\frac{x^2-1}{a-b}\times\frac{(a-b)^2}{x-1}$

C. **The quotient of one fraction divided by a second** is the product of the first and the reciprocal of the second.

Divide the following:

1. $\frac{11}{20}\div\frac{33}{25}$
2. $\frac{3a^2b}{8b^3}\div 6ab$
3. $\frac{a^2-ab}{16ab^2}\div\frac{a-b}{2a}$

421. For linear equations:

Illustration 1. Solve for x: $\frac{3x+1}{4}-2=4-\frac{x+1}{3}$

Multiply both sides by the L. C. M. of the denominators . . . $3(3x+1)-24=48-4(x+1)$

Performing multiplications . $9x+3-24=48-4x-4$

Combining terms . . . . $9x-21=44-4x$

Add $+21$ and $+4x$ to each side . $13x=65$

Divide both sides by 13 . . . . $x=5$

422. For quadratic equations:

Illustration 1. Solve for x: $3x^2-5x=7$ . . . (1)

Divide both sides by 3 . . . $x^2-\frac{5x}{3}=\frac{7}{3}$ . . . (2)

Add the square of $(\frac{1}{2}\cdot\frac{5}{3})$ to each side . . . . . $x^2-\frac{5x}{3}+(\frac{1}{2}\cdot\frac{5}{3})^2=\frac{7}{3}+\frac{25}{36}$. (3)

Take the square root of each side of the equation . . $x-\frac{5}{6}=\pm\sqrt{\frac{109}{36}}=\pm\frac{1}{6}\sqrt{109}$ . . (4)

$$x-\frac{5}{6}=\pm\frac{10.44}{6} \quad . \quad . \quad . \quad (5)$$

$$x=\frac{5}{6}+\frac{10.44}{6} \qquad x=\frac{5}{6}-\frac{10.44}{6}$$

$$=\frac{15.44}{6} \qquad =-\frac{5.44}{6}$$

$$=2.57+ \qquad =-.90+$$

Notice in (3), $(\frac{1}{2}\cdot\frac{5}{3})^2$ is added to the left side to make the left side a perfect square. It is added to the right side to preserve the balance of the equation. $(\frac{1}{2}\cdot\frac{5}{3})^2$, or $\frac{25}{36}$, is obtained by squaring half the coefficient of x. Notice that in step (2) the equation is divided by 3 to make the first term x^2, which is a perfect square.

Illustration 2. Solve for x:

$$x^2-x=20$$
$$x^2-x-20=0$$
$$(x-5)(x+4)=0$$
$$x-5=0 \qquad x+4=0$$
$$x=5 \qquad x=-4$$

Notice that a quadratic equation can always be solved as in Illustration 1, though often not so simply. Use the method given in Illustration 2 when convenient. It is easier.

Solve the following equations:

1. $\frac{x-2}{3}=\frac{x-4}{5}$
2. $\frac{2x-7}{7}-\frac{3+2x}{4}=2$
3. $\frac{4}{x-5}=\frac{3}{2x+7}$
4. $\frac{3-4x}{8}-\frac{5-2x}{3}=1-\frac{x}{4}$
5. $x^2+3x=18$
6. $2x^2-x=15$
7. $3x^2-11x=2$
8. $2x^2+5x=17$

423. To solve a system of equations consisting of two equations containing two unknowns, eliminate one of the unknowns and solve the resulting equation for the other.

A. **When both equations are of the first degree,** eliminate by addition or subtraction.

Illustration 1. Solve for x and y: $\begin{cases}5x-4y=6.5\\7x+5y=38.25\end{cases}$

(1) $5x-4y=6.5$
(2) $7x+5y=38.25$
(1)$\times 7$ $35x-28y=45.5$
(2)$\times 5$ $35x+25y=191.25$
$-53y=-145.75$ ..... Subtract the fourth
$y=2.75$ equation from the third

Notice that x may be found by multiplying equation (1) by 5 and equation (2) by 4 and adding the results or by substituting 2.75 for y in either equation (1) or (2) and solving the result for x.

B. **When one equation is of the first degree and one of the second,** eliminate by substitution.

Illustration 2. Solve for x and y: $\begin{cases} x+y=8 & . \quad . \quad (1) \\ x^2+y^2=34 & . \quad . \quad (2) \end{cases}$

Solve (1) for x, $\qquad x=8-y \quad . \quad (3)$

Substitute $8-y$ for x in (2) $\quad (8-y)^2+y^2=34$

$$64-16y+2y^2=34$$
$$2y^2-16y+30=0$$
$$y^2-8y+15=0$$
$$(y-5)(y-3)=0$$
$$y=5 \text{ and } y=3$$

To find x, substitute the values of y in (3):

$y=5$	$y=3$
$x=8-y$	$x=8-y$
$=8-5$	$=8-3$
$=3$	$=5$

The solutions are $\begin{cases} x=3 \\ y=5 \end{cases}$ $\quad$ $\begin{cases} x=5 \\ y=3 \end{cases}$

Solve the following systems for x and y:

1. $2x+3y=16$, $5x-2y=21$
2. $5x-2y=14$, $xy=20$
3. $2x^2+y^2=57$, $x-y=1$

III. TABLES

TABLE OF SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9
0	0.000	1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	7.071	7.141	7.211	7.280	7.349	7.416	7.483	7.550	7.616	7.681
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.900	9.950

TABLE OF SINES, COSINES, AND TANGENTS

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.017	1.000	.017	46	.719	.695	1.036
2	.035	.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.087	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.469	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.423	2.145
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.424	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.748
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.469	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.332
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.430
41	.656	.755	.869	86	.998	.070	14.301
42	.669	.743	.900	87	.999	.052	19.081
43	.682	.731	.933	88	.999	.035	28.636
44	.695	.719	.966	89	1.000	.017	57.290
45	.707	.707	1.000				

UNITS OF LENGTH

English

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards = 1 rod (rd.)
320 rods or 5280 feet = 1 mile (mi.)

Metric

10 centimeters (cm.) = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
1000 meters = 1 kilometer (km.)
1 meter = 39.37 inches
1 kilometer = .62 of a mile
1 foot = 30.48 centimeters
1 mile = 1.6093 kilometers

UNITS OF SURFACE

English

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
4840 square yards = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)

Metric

100 square centimeters = 1 square decimeter
100 square decimeters = 1 square meter

IV. The Assumptions

A. The Axioms

Ax. 1. If equals are added to equals, the results are equal. p. 29, p. 328

Ax. 2. If equals are subtracted from equals, the results are equal. . . . . p. 29, p. 328

Ax. 3. Doubles of equals are equal, or, more generally, if equals are multiplied by equals, the results are equal. . . . . p. 30

Ax. 4. Halves of equals are equal, or, more generally, if equals are divided by equals, the results are equal. . . . . p. 30

Ax. 5. Quantities equal to the same or to equal quantities are equal to each other. . . . . p. 30

Ax. 6. Equals may be substituted for equals in statements of equality. . . . . p. 30

Ax. 7. The whole is greater than any of its parts. . . p. 82, p. 163

Ax. 8. If the first quantity is greater than the second and the second is greater than or equal to the third, then the first is greater than the third. . . . . p. 82, p. 163

Ax. 9. If equals are added to unequals, the results are unequal in the same order. . . . . p. 162

Ax. 10. If equals are subtracted from unequals, the results are unequal in the same order. . . . . p. 162

Ax. 11. If unequals are added to unequals, the greater to the greater and the lesser to the lesser, the results are unequal in the same order. . . . . p. 162

Ax. 12. If unequals are subtracted from equals, the results are unequal in the opposite order. . . . . p. 163

Ax. 13. If unequals are multiplied by the same positive number, the results are unequal in the same order. . . . . p. 163

Ax. 14. If unequals are divided by the same positive number, the results are unequal in the same order. . . . . p. 163

Ax. 15. Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio. . . . . p. 254

B. The Postulates

Post. 1. An indefinite number of straight lines can be drawn through one point. . . . . p. 19

Post. 2. Only one straight line can be drawn through two given points; that is, two given points locate a straight line definitely or determine a straight line. . . . . p. 19

Post. 3. Two different straight lines can intersect in only one point; that is, two intersecting straight lines locate or determine a point. . . . . p. 19

Post. 4. Only one segment can be drawn between two given points; that is, a segment is determined in position and in length if its end points are known. . . . . p. 19

Post. 5. A segment can have one and only one mid-point. . . . p. 19

Post. 6. Only one ray can be drawn from a given origin and pass through a second given point. . . . . p. 19

Post. 7. Any figure may be moved about in space without changing its size or shape. . . . . p. 31

Post. 8. Figures congruent to the same figure are congruent to each other. . . . . p. 31

Post. 9. An angle can be bisected by one and only one line. . p. 31

Post. 10. One perpendicular and only one can be drawn to a line through a point in the line. . . . . p. 31

Post. 11. One perpendicular and only one can be drawn to a line from a point not in the line. . . . . p. 31

Post. 12. The perpendicular is the shortest segment from a point to a straight line. . . . . p. 31

Post. 13. All straight angles are equal . . . . . p. 31

Post. 14. All right angles are equal. . . . . p. 31

Post. 15. Complements of the same or of equal angles are equal. p. 31

Post. 16. Supplements of the same or of equal angles are equal. . . . . p. 31

Post. 17. Vertical or opposite angles are equal. . . . . p. 31

Post. 18. If a ray starts from a point in a straight line, the sum of the two adjacent angles formed is a straight angle, or 180°. . . . p. 31

Post. 19. If any number of rays start from a point in a straight line and extend on the same side of that line, the sum of the adjacent angles formed is a straight angle, or 180°. . . . . p. 32

Post. 20. If two supplementary angles are adjacent, their exterior sides lie in a straight line.....p. 32

Post. 21. Radii of the same circle are equal.....p. 45

Post. 22. Only one line can be drawn through a given point parallel to a given line, or two lines that intersect cannot both be parallel to the same line.....p. 84

Post. 23. The sum of two sides of a triangle is greater than the third.....p. 163

Post. 24. The difference between two sides of a triangle is less than the third.....p. 163

Post. 25. The diameter of a circle is twice its radius.....p. 192

Post. 26. A circle is located definitely if its center and its radius are known.....p. 192

Post. 27. Circles with equal radii are congruent.....p. 192

Post. 28. Congruent circles have equal radii.....p. 192

Post. 29. Every diameter bisects the circle.....p. 192

Post. 30. Between the same two points on a circle there is one and only one minor arc, provided these two points are not the ends of a diameter.....p. 193

Post. 31. In the same circle or in congruent circles, equal central angles intercept equal minor arcs.....p. 193

Post. 32. In the same circle or in congruent circles, equal minor arcs intercept equal central angles.....p. 193

Post. 33. If a line passes through a point within a circle, the line and the circle intersect in two and only two points.....p. 193

Post. 34. A segment joining a point without a circle and the center is longer than the radius.....p. 193

Post. 35. If a segment that has one end at the center of a circle is longer than the radius, it extends without the circle and cuts the circle but once.....p. 193

Post. 36. A segment joining a point within a circle and the center is shorter than the radius.....p. 193

Post. 37. If a segment that has one end at the center of a circle is shorter than the radius, it lies wholly within the circle..p. 193

Post. 38. Through three given points not in the same straight line one circle and only one can be drawn.....p. 197

Post. 39. Two circles cannot intersect in more than two points.p. 198

Post. 40. In the same circle or in congruent circles, if two central angles are unequal, the minor arc intercepted by the greater angle is greater than the minor arc intercepted by the lesser angle. . . . . p. 200

Post. 41. In the same circle or in congruent circles, if two minor arcs are unequal, the angle intercepted by the greater arc is greater than the angle intercepted by the lesser arc. . . . . p. 200

Post. 42. The measure number of a central angle and the measure number of its intercepted arc are the same, or a central angle is measured by its arc. . . . . p. 223

Post. 43. In the same circle or in congruent circles, angles are equal if they are measured by equal arcs . . . . . p. 229

Post. 44. In the same circle or in congruent circles, arcs are equal if they measure equal angles . . . . . p. 229

Post. 45. If two equal segments begin at the same point and extend along the same line in the same direction, they end at the same point. . . . . p. 267

Post. 46. If congruent polygons are added to or subtracted from congruent polygons, the results are polygons that cover the same extent of surface. . . . . p. 311

Post. 47. If series of regular polygons are inscribed in and circumscribed about the same circle (radius r) and the number of sides is increased indefinitely, the limit of the perimeters of these series is the same. This limit is $2\pi r$. . . . . p. 396

Post. 48. If series of regular polygons are inscribed in and circumscribed about the same circle (radius r) and the number of sides is increased indefinitely, the limit of the areas of these series is the same. This limit is one-half the product of the radius of the circle and its circumference . . . . . p. 397

Post. 49. The area of a sector has the same ratio to the area of a circle of which it is a part as the angle of the sector has to four right angles. . . . . p. 397

V. Summary of Equal Segments and Angles, Parallels, and Perpendiculars

Equal segments are formed by:

1. Segments equal to the same or to equal segments. . . Ax. 5, p. 30
2. Sums, differences, equal multiples, or equal parts of equal segments. . . . . Axs. 1–4, pp. 29, 30
3. Radii of the same or equal circles. . Post 21, p. 45; Post. 28, p. 192
4. Sides of an isosceles triangle. . . . . §60, p. 44
5. Corresponding sides of congruent triangles. . . . . §76, p. 52
6. Opposite sides of a parallelogram. . . . . §149, p. 132
7. Parallel or perpendicular segments between parallels . . . . . §§150, 151, p. 132
8. Sides of a rhombus or square. . . . . §152, p. 133
9. The intersection of the diagonals of a parallelogram. Th. 25, p. 134
10. The diagonals of a rectangle. . . . . Ex. 2, p. 134
11. The non-parallel sides of an isosceles trapezoid. . . . . §143, p. 129
12. A series of parallels cutting equal segments on one transversal and intersecting a second. . . . . Th. 28, p. 142
13. A segment parallel to the base of a triangle and bisecting one side. . . . . §163, p. 143
14. Segments joining a point in the perpendicular bisector of a segment to the ends of a segment. . . . . §190, p. 181
15. Perpendiculars drawn from a point in the bisector of an angle to the sides of the angle. . . . . §193, p. 183
16. A line through the center of a circle perpendicular to a chord. . . . . Th. 42, p. 196
17. The line of centers of two circles intersecting the common chord. . . . . §207, Ex. 1, p. 194

18. A common chord intersecting the line of centers of two congruent circles. . . . . §207, Ex. 2, p. 194
19. Chords with equal central angles in the same circle or in congruent circles. . . . . Th. 40B, p. 194
20. Chords with equal arcs in the same circle or in congruent circles . . . . . Th. 41B, p. 195
21. Chords equally distant from the center in the same circle or in congruent circles. . . . . Th. 43, p. 199
22. Perpendiculars from the center to two equal chords in the same circle or in congruent circles. . . . . Th. 44, p. 199
23. Tangents to a circle from a point without the circle. Th. 53, p. 210
24. The fourth terms in two proportions that have three terms of one equal to three corresponding terms of another. . . . . §264, p. 255
25. Sides of a regular polygon. . . . . §351, p. 368

Two angles are equal if they are:

1. Angles equal to the same or to equal angles. . . . . Ax. 5, p. 31
2. Sums, differences, equal multiples, or equal parts of equal angles. . . . . Axs. 1–4, pp. 29, 30
3. Right angles or straight angles. . . . . Posts. 13, 14, p. 31
4. Supplements or complements of the same or of equal angles. . . . . Posts. 15, 16, p. 31
5. Vertical angles. . . . . Post. 17, p. 31
6. Corresponding angles of congruent triangles. . . . . §76, p. 52
7. Base angles of an isosceles triangle. . . . . Th. 3, p. 57
8. Alternate interior angles of parallel lines. . . . . Th. 12, p. 93
9. Corresponding angles of parallel lines. . . . . Th. 13, p. 94
10. The third angles of triangles that have two angles of one equal respectively to two angles of the other. . §122, p. 104

11. Angles with their sides parallel right side to right side. . . . . Th. 16, p. 99
‡12. Angles with their sides perpendicular right side to right side. . . . . Th. 19, p. 110
13. Opposite angles of a parallelogram. . . . . §153, p. 133
‡14. Base angles of an isosceles trapezoid. . . . . B Ex. 1, p. 135
15. Central angles having equal minor arcs in the same circle or in congruent circles. . . . . Post. 32, p. 193
16. Central angles having equal chords in the same circle or in congruent circles. . . . . Th. 40A, p. 194
17. Measured by equal arcs in the same circle or in congruent circles. . . . . Post. 43, p. 229
18. Corresponding angles of similar figures. . . . . §291, p. 271
19. Angles of a regular polygon. . . . . §351, p. 368

Two straight lines in the same plane are parallel if:

1. The alternate interior angles are equal. . . . . Th. 8, p. 86
2. The corresponding angles are equal. . . . . Th. 9, p. 87
3. The interior angles on the same side of the transversal are supplements. . . . . Th. 10, p. 88
4. They are parallel to the same line . . . . . Th. 7, p. 85
5. They are perpendicular to the same line . . . . . Th. 11, p. 89
‡6. One is one side of a triangle and the other passes through the mid-points of the other two sides. . . . . Th. 29, p. 144
7. One is one side of a triangle and the other divides the other two sides proportionally. . . . . Th. 62, p. 266

Right angles and perpendiculars are formed when:

1. One line passes through a point in another so that the adjacent angles formed are equal. . . . . §36, p. 22
2. One line is perpendicular to a line parallel to another. Th. 15, p. 96
3. One line passes through two points each equally distant from the ends of a segment of another. . . . . Th. 5, p. 73
4. Any two circles intersect, and one is the line of centers and the other the common chord. . . . . §207, Ex., p. 194
5. An angle is inscribed in a semicircle. . . . . §247, p. 225
6. A radius is drawn to the point of contact of a tangent Th. 52, p. 206

‡ Optional.

VI. Formulas Used in Plane Geometry

A. Notation

The following notation is used:

a, b, and c = sides of $\triangle ABC$, a opposite $\angle A$, etc.
$s = \frac{1}{2}(a+b+c)$
In a right triangle $\angle C = 90°$
p = perimeter
p_b^a = projection of a on b
A = area
r = radius of a circle
c = circumference of a circle
d = diameter of a circle or diagonal of a square
h = altitude
h_a = altitude on side a
b = base of a triangle or parallelogram
b and b' = the bases of a trapezoid
m_a = median of a triangle on side a
t_A = bisector of $\angle A$
a = apothem of a regular polygon
$\pi = 3\frac{1}{7}$ or 3.142 or 3.1416 according to degree of accuracy required

B. Lengths of Lines

1. In an equilateral triangle: $h = \frac{b}{2}\sqrt{3}$; side $= \frac{2}{3}h\sqrt{3}$ ..... p. 294

2. In a square: $d = b\sqrt{2}$ ..... p. 294

3. In similar polygons: $\frac{a}{a'} = \frac{b}{b'}$ and $\frac{p}{p'} = \frac{a}{a'}$ ..... p. 274, p. 350

4. In circles: $\frac{c}{c'} = \frac{r}{r'} = \frac{d}{d'}$ ..... p. 398

5. In circles: $c = \pi d = 2\pi r$ ..... p. 396

6. The arc of a circle $= \frac{\text{central angle}}{360} \cdot 2\pi r$

7. $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ ..... p. 418

8. $t_A = \frac{2}{b+c}\sqrt{bcs(s-a)}$ ..... p. 417

9. If a circle is inscribed in $\triangle ABC$, $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ ... p. 420

10. If a circle is circumscribed about $\triangle ABC$,
$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$ ..... p. 419

11. In a triangle with $\angle C$ acute: $c^2 = a^2+b^2-2bp_b^a$ ..... p. 412

12. In a right triangle: $c^2=a^2+b^2$ ..... p. 289
13. In a triangle with $\angle C$ obtuse: $c^2=a^2+b^2+2bp_b^a$ ..... p. 413
14. One side of a regular inscribed polygon of $2n$ sides is $\sqrt{2r^2-r\sqrt{4r^2-a^2}}$, where a is one side of a polygon of n sides. p. 421
15. One side of a regular circumscribed polygon of $2n$ sides is $\frac{2ar}{2r+\sqrt{4r^2+a^2}}$, where a is one side of a polygon of n sides. p. 423

C. Areas of Plane Figures

1. For rectangles: $A=bh$ ..... p. 313, p. 317
2. For squares: $A=b^2$
3. For parallelograms: $A=bh$ ..... p. 316, p. 317
4. For triangles: $A=\frac{1}{2}bh$ ..... p. 317
5. For triangles: $A=\sqrt{s(s-a)(s-b)(s-c)}$ ..... p. 415
6. For equilateral triangles: $A=\frac{b^2}{4}\sqrt{3}$ ..... p. 325
7. For right triangles with hypotenuse as base: $A=\frac{b^2}{4}$ ..... p. 325
8. For trapezoids: $A=\frac{h}{2}(b+b')$ ..... p. 318
9. For regular polygons: $A=\frac{1}{2}ap$ ..... p. 376
10. For circles: $A=\pi r^2$ ..... p. 397
11. For sectors of circles: $A=arc\cdot\frac{1}{2}r=\frac{\text{central angle}}{360}\cdot\pi r^2$ ..... p. 397
12. For segment of circles: $A=$ sector $-\triangle$ formed by chord and two radii ..... p. 397
13. In any two similar polygons: $\frac{A}{A'}=\frac{a^2}{a'^2}$ ..... p. 352
14. In any two circles: $\frac{A}{A'}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$ ..... p. 398

GEOMETRIC REASONING

Clear thinking. Attention has already been called to the part the facts of geometry play in various fields of activity. Here we wish to emphasize the point that the methods of thinking employed in geometry are the same as those applied in many thinking situations in life. Indeed, geometry affords an illustration of how all careful thinking is done. The characteristics of such mental activity are brought out more clearly in geometry than elsewhere because of the simplicity of the subject matter.

In geometry we had to distinguish carefully between definitions, statements that were taken for granted without proof such as axioms and postulates, and statements that were not taken for granted but were to be proved. The same thing is true in all reasoning.

Definition of terms. Terms used must be clearly defined so that meanings are understood. In geometry it was necessary to define parallel lines, for example, before we began our study of such lines on page 82, and to define similar triangles before we could study such triangles. In fact, our study of parallel lines and of similar triangles was based directly on these definitions.

It is fairly easy to define the terms used in geometry. In other subjects it is not always so simple. If we are to count the houses in a block, we must know what a house is. Is a bird house one of them? A real estate man talks about an eight-room or a ten-room house. He must have a definite idea of what a room is. Does he count a large closet, a coal cellar, or a bathroom? The census taker must set down the number of farms in his district. He must have a clear conception of a farm from which to work. Is a three-acre lot a farm? A

librarian has to report the number of books in his library. He must know what a book is. Is a two-page pamphlet a book? In all these cases it is just as necessary to have clear-cut definitions on which to base the work as it was to be able to define parallel lines or similar triangles before beginning the study of parallel lines or similar triangles.

In geometry there are some terms we do not try to define, such as *point*, *line*, and *straight*. So in other subjects there are terms that cannot be defined, such as *justice*, *liberty*, *beauty*, etc. We have here an illustration of the fact that geometric thinking is really simple as compared with other thinking. When we talk about a triangle in a geometry class, everyone has the same thing in mind; but when a group of people are talking about liberty and justice, no two of them have exactly the same thing in mind, and the discussion often results in much confusion of thought and in hard feeling. Occasionally someone in a geometry class is badly "mixed up" for several days because he has failed to comprehend a necessary definition.

Reasoning from assumptions. Much of the work in circles is often more difficult than need be because the pupil does not know the postulates on which the proofs are based. Could we prove Theorem 51, page 205, for example, if we did not know Postulates 12 and 31?

A large part of modern life is based on certain facts that everyone takes for granted. Every time we post a letter, we assume that the United States mail is trustworthy. Every time we accept paper money, we assume that because it was issued by the United States government other people will accept it for face value. The paper itself is worthless except as a stable and honorable government is behind it.

There are many postulates on which our modern life is based which may or may not be true in fact, but which must be recognized if our life is to go on as it does. Most of them cannot be proved. A retail merchant leaves a large part of his goods within easy reach on his counters and shelves because he finds it good business to assume that people are honest. Most school examinations are based on the assumption that pupils will not cheat when placed on their honor. The rules of a game are arbitrary assumptions. The usages of good society are assumptions; for example, it is an arbitrary assumption of good table manners that we should not put food into our mouths with our knives. In some parts of Arabia it is equally good table manners to use fingers for such a purpose. In debate the best point of attack is often the assumptions that the opponent has made.

Successful discussion is possible only when people have a clear understanding of the meaning of the terms used and base their thinking on the same assumptions.

Proving statements true. Perhaps the most vital point in geometric reasoning is showing that a certain statement is necessarily true if certain other statements are true. This we have called proof. "If A is greater than B and B is greater than C, then A is greater than C." This form of thinking is found in all reasoning. "If John is the father of James and James is the father of Thomas, then John is the grandfather of Thomas." "A registered voter has a right to vote. I am a registered voter; therefore I have a right to vote." There is no getting away from such conclusions. They must follow.

Drawing conclusions. Much of our mental activity in everyday life, however, is not of this kind. Instead, it may be said that we "jump at conclusions." Someone may say that if he takes a certain medicine he will get

well. Such a conclusion does not necessarily follow. All that may be said is that he may get well. A preponderance of evidence in favor of the medicine may make it probable that he will get well, but no amount of evidence will make it certain that he will. We hear many such statements every day. "If I vote for this man, he will be elected"; "If I live in New York, I am a citizen of the United States"; "If water is cooled to 32 degrees F., it will freeze," are examples. The drawing of correct conclusions is a rather complicated matter. It is carefully considered in the study of logic.

Indirect proof. Many theorems in geometry are proved by what is known as the indirect method. (See Theorem 7, page 85.) This is a very common method of proof in many lines of thought. The following are illustrations of indirect proof:

An alibi is proved. John is accused of taking a watch from a certain house between ten and eleven o'clock on the evening of June 20. Either John took the watch or he did not. Assume that John did take the watch. If so, he must have been in the house at that time. If, however, John can prove that he was twelve miles away from the house on the night and at the hour in question, we have arrived at a contradiction, and our assumption that John took the watch is evidently not true. Of course, an argument of this kind does not take into account the fact that John may have had someone else get the watch for him, but it is evident that John himself did not do it.

A certain book is said to be two hundred years old. Either it is two hundred years old or it is not. Assume that the book is two hundred years old. Then it could not refer to events that happened since 1732. But on page 95 is a reference to the Battle of Waterloo, which

was fought in 1815. Therefore our assumption that the book is two hundred years old is not true.

A certain box is said to have been made before the War of the Revolution. Either the box was made before 1776 or it was not. Assume that the box was made before that date. Then it could not have been put together with wire nails, for wire nails were not made until long after that date. But the box is put together with wire nails. Therefore the box is not as old as is asserted.

The indirect method of proof is frequently used in showing up fake antiques and forged paintings, and is exactly like our proof for Theorem 7.

This method is very helpful in cases in which there are many possibilities. Often the difficulty lies in listing them all. But even if they cannot all be listed, the method is still of value, for some of the possibilities may be eliminated by being proved untrue.

A desk lamp fails to light when the button is pressed. The trouble may be in the bulb, in the cord connecting the lamp with the socket, in the socket, in the circuit, in the house fuse, in the transmission line, or in the power house. Other lights in the same room will work, so the last three possibilities are eliminated. A new bulb is tried; the trouble is not in the bulb. It is traced until it is found to be in the cord connecting the lamp with the socket.

Physicians, automobile mechanics, and many others constantly employ such methods.

Synthetic proof. Most of the theorems in geometry are proved by a chain of reasoning. Sometimes such a chain is made to proceed from start to finish. It is then called a synthetic proof. Numerous discoveries and inventions have been made in this way. The following are examples of this method:

What kind of tree is that in the next yard? Its leaves stay green all winter. It is an evergreen. The leaves are long and pointed like needles. This shows that it belongs to the cone-bearing evergreens. The leaves are in bunches. It is therefore a pine. There are five leaves in each bunch. It is therefore a white pine.

Here is a book to be cataloged and placed on the shelves of a library. It is a book on mathematics. Its number will be between 510 and 520. It is a geometry. The number is 513. It is a solid geometry. Its number is then 513.3.

The attendant at an automobile club is asked for a routing from Peoria, Illinois, to Augusta, Maine. He says, "You must go east. Take Route 24 out of Peoria to Napoleon, Ohio, then 34 to Sandusky, then 29 to Buffalo, Albany, and Boston, except for a detour or two [which he points out], then over Route 1 to Portland, Maine, and 201 to Augusta." His thinking is from start to finish, as in the other two foregoing illustrations.

Analysis. Another method of forging one of these chains of reasoning has been emphasized in this book. If the final conclusion is known, one may work back from the conclusion. This method is called analysis. The following are examples:

A merchant must be at Jonesboro at 8:00 P.M. on July 7. He asks the ticket agent how to get there by rail on time. The agent finds Jonesboro in the official railroad guide. He tells the man that he can reach Jonesboro at 5:00 P.M. by train on the L. & T. leaving Julesburg at 4:00 P.M. He can reach Julesburg on the P., T. & K. at 3:30 by leaving Chicago at 12:00 noon. He can reach Chicago at 11:00 A.M. on the N. Y. C. by leaving his home town at 9:30 A.M. In this way the ticket agent traces connections from the finish to the

start, and the merchant knows that he must leave home at 9:15 in the morning to fill his appointment at Jonesboro.

A lawyer used the analytic method in a certain case. A court had decided against his client. The client had lost what is called his equity in a piece of property. The lawyer started with the idea that the client should still have a real right to the property. Step by step he traced the matter back through decision after decision until he found the point where the right seemed to have been lost. He then built up his argument and won a case which had been carried through twenty years of litigation.

The method of reasoning back from the conclusion can be used only when the conclusion is known or when one can make a reasonable guess at a conclusion. This is what the scientific man often does. He makes a guess and then works back from his guess. If it fits in with the known facts, well and good. If not, he must alter his guess or throw it away and make a new one.

Value of geometric reasoning. Your experience in reasoning in geometry will not make you a careful thinker in all fields of learning, but it should help you to realize the importance of clear thinking. You should be a more logical thinker because you have had this experience.

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